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Convolution semigroups

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Convolution Semigroups

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Dedication

This dissertation is dedicated
to the memories of

Carl Wayne “Bubba” Belello,
Tony Wayne Zito,
& Frank Zito

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Table of Contents

List of Figures

Abstract

In this dissertation we investigate, compute, and approximate convolution powers of functions (often probability densities) with compact support in the positive real numbers. Extending results of Ursula Westphal from 1974 concerning the characteristic function on the interval $[0,1]$, it is shown that positive, decreasing step functions with compact support can be embedded in a convolution semigroup in $L^1(0,\infty)$ and that any decreasing, positive function $p \in L^1(0,\infty)$ can be embedded in a convolution semigroup of distributions. As an application to the study of evolution equations, we consider an evolutionary system that is described by a bounded, strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in combination with a probability density function $p \in L^1(0,\infty)$ describing when an observation of the system is being made. Then the n^{th} convolution power $p^{\star n}$ of p is the probability distribution describing when the n^{th} observation of the system is being made and $E_n(x_0) := \int_0^\infty T(s)x_0\, p^{\star n}(s)\, ds$ is the expected state of the system at the n^{th} observation. We discuss approximation procedures of $E_n(x_0)$ based on approximations of the semigroup T (in terms of its generator A) and of $p^{\star n}$ (in terms of its Laplace transform $\widehat{p}$).

Introduction

Let X be a Banach space containing the states x of a system that evolves in time. Let $x_0 \in X$ denote a possible initial state of the system at time $t = 0$. We assume that, for each $t \geq 0$, there is a bounded linear operator $T(t) : X \to X$ such that $T(t)x_0$ describes the state of the system at time $t \geq 0$. In particular, we assume that $T(0)x_0 = x_0$; i.e.,

$$T(0) = I. \tag{I}$$

We further assume that the system is deterministic and does not change by being observed. This implies the following. Let $x_1 = T(t_1)x_0$ describe the state of the system (that started with initial state given by x_0) after t_1 time units have passed. Assume that two people John One and John Two observe the system where John One starts observing the system at time $t = 0$ and John Two starts observing at time $t = t_1$. After an additional $t_2 > 0$ time units, John One will observe the state

$$y_1 = T(t_1 + t_2)x_0,$$

whereas John Two observes

$$y_2 = T(t_2)T(t_1)x_0.$$

Now since John One and John Two observe a deterministic system that does not change by being observed, it must follow that $y_1 = y_2$; i.e., for all $t_1, t_2 \geq 0$ and all $x_0 \in X$, $T(t_2 + t_1)x_0 = T(t_2)T(t_1)x_0$ or

$$T(t_2 + t_1) = T(t_2)T(t_1). \tag{II}$$

Lastly, we assume that the map

$$t \mapsto T(t)x_0 \tag{III}$$

is continuous (from $[0, \infty)$ into X) for all $x_0 \in X$ and that there exists a constant $M > 0$ such that

$$\|T(t)x_0\| \leq M\|x_0\| \tag{IV}$$

for all $x_0 \in X$ and $t \geq 0$. We denote by $L(X)$ the Banach space of all bounded linear operators on X, and we call a family $\{T(t)\}_{t\geq 0} \subset L(X)$ a "bounded strongly continuous semigroup" if it satisfies conditions (I)-(IV). The generator of a strongly continuous semigroup is given by

$$A : x \mapsto \lim_{t\to 0} \frac{T(t)x - x}{t},$$

with domain $D(A)$ consisting of those $x \in X$ for which the limit exists. Then it is well known (see [1]) that $u(t) := T(t)x_0$ is the solution of the "abstract Cauchy problem"

$$(ACP) \quad \begin{cases} u'(t) = Au(t) \qquad (t \geq 0) \\ u(0) = x_0. \end{cases} \tag{1}$$

That is, (ACP) is the underlying differential equation (initial value problem) describing the evolution of the system under consideration.

Let $0 \leq p \subset L^1(0,\infty)$ be a probability density describing when an observation of the system is made; i.e., if

$$P(u) := \int_0^u p(s)\,ds$$

denotes the associated probability distribution, then, for $t_2 \geq t_1 \geq 0$,

$$P(t_2) - P(t_1) = \int_{t_1}^{t_2} p(s)\,ds$$

is the probability that the system is being observed between time t_1 and time t_2 and

$$E_1(x_0) := \int_0^\infty T(s)x_0\, p(s)\,ds$$

is the expected state of the system when it is being observed. If the observations are made independently (i.e., the time it takes to make the first observation does not affect the time it takes to make the second observation), then the convolution square

$$p^{\star 2}(u) = (p \star p)(u) := \int_0^u p(u-s)p(s)\,ds$$

gives the probability density describing when a second observation is being made, and

$$E_2(x_0) := \int_0^\infty T(s)x_0\,p^{\star 2}(s)\,ds$$

is the expected state of the system at the second observation. In general, for $n \in \mathbb{N}$ $p^{\star n} := p^{\star(n-1)} \star p$ is the probability density describing when the system is observed for the n^{th} time and

$$E_n(x_0) := \int_0^\infty T(s)x_0\,p^{\star n}(s)\,ds$$

is the expected state of the system when it is observed for the n^{th} time. It is the purpose of this dissertation to discuss the computation (approximation) of the expected states $E_n(x_0)$ given a semigroup $\{T(t)\}_{t\geq 0}$ and a probability density $0 \leq p \in L^1(0,\infty)$. This leads to the following structure of the dissertation.

Firstly, we will discuss how to compute the convolution powers $p^{\star n}$ given a probability density

$$0 \leq p \in L^1(0,\infty).$$

Out of pure mathematical curiosity, we will take on a more general task and ask if there is a family $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$ such that $p_1 = p$ and $p_{\alpha+\beta} = p_\alpha \star p_\beta$ for all $\alpha, \beta > 0$ and, if so, how can it be computed (approximated) given its generating function p. Secondly, we will discuss the approximation of the expected values $E_n(x)$ given $0 \leq p \in L^1(0,\infty)$ and given an operator A with domain and range in X that describes how the system under consideration changes in time (i.e., given the "generator" A of the semigroup $\{T(t)\}_{t>\geq 0}$). In Chapter 1, we discuss the convolution semigroups $\{p_\alpha\}_{\alpha>0}$ generated by

(a) $p(u) = \chi_{[0,1]}(u)$,

(b) $p(u) = a\chi_{[0,1/2]}(u) + b\chi_{(1/2,1]}(u)$, or

(c) $p(u) = \sum_{i=0}^{n-1} a_i \chi_{(i/n,(i+1)/n]}(u)$.

Extending results by Ursula Westphal [30] from 1974, we show that if the step functions p are (i) positive and (ii) decreasing, then there exists a convolution semigroup $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$ with $p_1 = p$.

Clearly, if $0 \leq p$ is a probability density with support in $[0,1]$ (describing the probability when an observation of the system $T(t)x$ is being made during the time interval $[0,1]$), then $0 \leq p^{\star n} = p_n$ is a probability density will support in $[0,n]$ (describing the probability when the n^{th} observation of the system $T(t)x$ is being made). However, as it will turn out, for $\alpha \notin \mathbb{N}$, the functions $p_\alpha \in L^1(0,\infty)$ are, in general, no longer positive (i.e., they are not probability densities), and they have non-compact support. Since all positive, decreasing step functions with compact support can be embedded into a convolution semigroup $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$, it can be expected that this holds true for all positive, decreasing $p \in L^1(0,\infty)$ with compact support. In Section 1.4 a slightly weaker statement will be proved. Using the concept of "log admissibility" of the Laplace transform $\widehat{p}(z) := \int_0^\infty e^{-zu} p(u)\, du$, we show that every positive and decreasing p that is continuous and has compact support can be embedded in a convolution semigroup $\{p_\alpha\}_{\alpha>0}$ of distributions. To numerically approximate p_α we use the fact that the Laplace transform maps convolution

$$(f \star g)(u) := \int_0^u f(u-s) g(s)\, ds \tag{2}$$

into multiplication. That is, if $f, g \in L^1(0,\infty)$, then $f \star g \in L^1(0,\infty)$, $\|f \star g\|_1 = \|f\|_1 \|g\|_1$, and

$$\widehat{(f \star g)}(z) = \widehat{f}(z) \widehat{g}(z) \tag{3}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$. In particular, if p is a probability distribution with support in $[0,1]$, then

$$\widehat{p}(z) = \int_0^\infty e^{-zu} p(u)\, du = \int_0^1 e^{-zu} p(u)\, du$$

and $(\widehat{p}(z))^\alpha$ is well defined for all $\alpha, z > 0$ since $\widehat{p}(z) > 0$ for all $z \in \mathbb{R}$.

If p can be embedded into a convolution semigroup $\{p_\alpha\}_{\alpha>0}$, then it follows from the convolution property $p_{\alpha+\beta} = p_\alpha \star p_\beta$ and $p_1 = p$ that

$$\widehat{p_\alpha}(z) = (\widehat{p}(z))^\alpha$$

for all $\mathrm{Re}(z) \geq 0$ and $\alpha > 0$ Thus, p_α is the inverse Laplace transform of $(\widehat{p})^\alpha$. In order to compute (approximate) the convolution powers p_α (and, in particular, the convolution powers $p_n = p^{\star n}$ for $n \in \mathbb{N}$) we use recently developed inversion formulas for the Laplace transform due to Patricio Jara, Frank Neubrander, and Koray Özer. Finally, in Chapter 2 we combine results concerning the approximation of

(a) $T(s)x$ (in terms of the generator A)

(b) p_α (in terms of the Laplace transform $\widehat{p}(z)$ of the generating probability function p)

in order to obtain approximations (with error estimates) for

$$E_\alpha(x) := \int_0^\infty T(s)x\, p_\alpha(s)\, ds,$$

the expected state of system when the α^{th} observation of it is being made.

Chapter 1
Non-Positive Convolution Semigroups

In this chapter we will discuss a method by which we may construct, for certain $0 \leq p \in L^1(0,\infty)$, a family of functions $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$ which satisfy $p_1 = p$, and, for all $\alpha, \beta > 0$,

$$p_{\alpha+\beta}(u) = (p_\alpha \star p_\beta)(u) := \int_0^u p_\alpha(u-s)p_\beta(s)ds \tag{1.1}$$

for all $u > 0$. The family $\{p_\alpha\}_{\alpha>0}$ is called the convolution semigroup generated by p, and we can think of p_α as the α^{th} convolution power of p. That is, $p_\alpha = p^{\star\alpha}$ for all $\alpha > 0$.

The Levy-Khintchine Theorem (see [12, p, 653], and [25]) is a classical result in the framework of probability distributions characterizing those probability distributions P with support in $[0,\infty)$ for which there exists a family of probability distributions P_α (i.e., increasing normalized functions of bounded variation) which satisfies $P_{\alpha+\beta}(u) = P_\alpha(u) \star P_\beta(u) =: \int_0^\xi P_\alpha(\xi - s)dP_\beta(s)ds$ and $P_1 = P$ (infinitely divisible probability distributions). However, this result does not apply to the examples of functions we will be studying in this chapter; i.e., we will be studying functions p which generate convolution semigroups $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$ where at least one p_α is not a positive function (and thus, $P_\alpha(u) := \int_0^u p_\alpha(s)\,ds$ is not increasing). To our knowledge, such non-positive convolution semigroups were not yet studied in the literature in great detail, and so it is the purpose of this chapter to add some basic insights in the nature of the problem and its mathematical challenges.

1.1 Preliminaries

To set the stage, let $p(u) := \chi_{[0,1]}(u)$. Then from the definition of convolution it immediately follows that

$$p^{\star 2}(u) := (p \star p)(u) = \begin{cases} 0 \quad \text{if } u \leq 0, \\ \int_0^u 1\,ds \quad \text{if } 0 < u \leq 1, \\ \int_{u-1}^1 1\,ds \quad \text{if } 1 < u \leq 2 \\ 0 \quad \text{if } u > 2. \end{cases} = \begin{cases} 0 \quad \text{if } u \leq 0, \\ u, ds \quad \text{if } 0 < u \leq 1, \\ 2-u \quad \text{if } 1 < u \leq 2 \\ 0 \quad \text{if } u > 2. \end{cases} \tag{1.2}$$

That is, $p^{\star 2}(u) = (1 - |u-1|)\chi_{[0,2]}$ (see Figure 1.1).

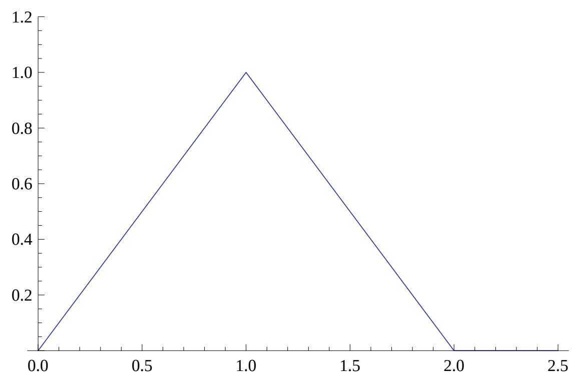


FIGURE 1.1. Graph of $p^{\star 2}$

Observe that the convolution of the characteristic function with itself is a continuous function with support in the interval $[0, 2]$. We will now construct the function $p^{\star 3} := u \mapsto \int_0^u p^{\star 2}(s)p(u-s)ds$. Let us first consider the case $u \leq 0$. Since $p^{\star 2}(s) = 0$ for all $s \leq 0$, it follows that $p^{\star 3}(u) = 0$ for $u \leq 0$. If $0 < u \leq 1$, then $p^{\star 3}(u) = \frac{u^2}{2}$. Moreover, if $1 < u \leq 2$, it follows that $p^{\star 3}(u) = \frac{-2u^2+6u-3}{2}$. Likewise, if $2 < u \leq 3$, then $p^{\star 3}(u) = \frac{u^2-6u+9}{2}$, and if $u > 3$, it follows that $p^{\star 3}(u) = 0$. We may then write the function $p^{\star 3}$ as

$$p^{\star 3}(u) = \begin{cases} 0 & \text{if } u < 0, \\ \frac{u^2}{2} & \text{if } 0 < u < 1, \\ \frac{-2u^2+6u-3}{2} & \text{if } 1 \leq u \leq 2, \\ \frac{u^2-6u+9}{2} & \text{if } 2 < u \leq 3, \\ 0 & \text{if } u > 3, \end{cases}$$

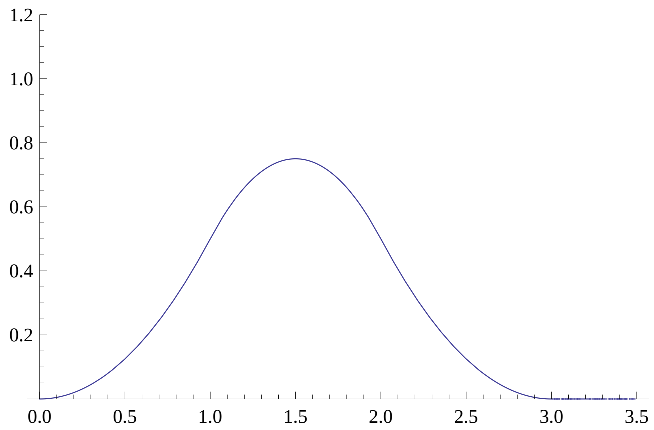


FIGURE 1.2. Graph of $p^{\star 3}$

Observe that the graph of $p^{\star 3}$, the third convolution power of the discontinuous function p, is once continuously differentiable (see also Proposition 1.2 (v) below). Though we have shown that it is possible to calculate convolution powers using the definition (albeit of an extremely simple function), we must note here two important points. Firstly, the process of computing convolution powers can be tedious. Secondly, and more importantly, we have not gotten any closer to our goal of embedding the characteristic function p into a convolution semigroup $\{p_\alpha\}_{\alpha>0}$. What follows is a well known property of the Laplace transform which guides us toward a solution to this problem (see Proposition 1.6.4 in [1]).

Let $f_1, f_2 \in L^1(0,\infty)$ and let $\widehat{f_1}$ and $\widehat{f_2}$ denote the Laplace transforms of f_1 and f_2 respectively. Then $(f_1 \star f_2) \in L^1(0,\infty)$, $\|f_1 \star f_2\|_1 = \|f_1\|_1\|f_2\|_2$, and

$$\widehat{f_1 \star f_2} = \widehat{f_1}\widehat{f_2}, \tag{1.3}$$

for all $\mathrm{Re}(z) \geq 0$.

To see how this result facilitates the task of computing convolution powers of $p(u) = \chi_{[0,1]}(u)$, observe that for $z \neq 0$,

$$\widehat{p}(z) := \int_0^\infty e^{-zu} p(u)\, du = \int_0^1 e^{-zu}\, du = \frac{1-e^{-z}}{z}. \tag{1.4}$$

Thus,

$$\widehat{p^{\star 2}}(z) = (\widehat{p}(z))^2 = \frac{1}{z^2} - \frac{2}{z^2}e^{-z} + \frac{1}{z^2}e^{-2z}. \tag{1.5}$$

Recall the following elementary operational rules from Laplace transform theory. If $\widehat{f}$ is the Laplace transform of a function $f \in L^1(0,\infty)$, then the function $z \mapsto e^{-az}\widehat{f}(z)$ is the Laplace transform of $u \mapsto f(u-a)\chi_{[a,\infty]}(u)$. Since the Laplace transform of $u \mapsto \frac{u^n}{n!}$ is $z \mapsto \frac{1}{z^{n+1}}$, it follows from (1.5) that

$$\begin{aligned} p^{\star 2}(u) &= u - 2(u-1)\chi_{[1,\infty]}(u) + (u-2)\chi_{[2,\infty]}(u) \\ &= (1 - |u-1|)\chi_{[0,2]}(u). \end{aligned}$$

In general, since

$$\widehat{p^{\star n}}(z) = (\widehat{p}(z))^n = \left(\frac{1-e^{-z}}{z}\right)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{1}{z^n} e^{-jz}, \tag{1.6}$$

it follows that

$$\begin{aligned} p^{\star n}(u) &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(u-j)^{n-1}}{(n-1)!} \chi_{[j,\infty]}(u) \\ &= \frac{1}{\Gamma(n)} \sum_{j=0}^{[u]} (-1)^j \binom{n}{j} (u-j)^{n-1}, \end{aligned} \tag{1.7}$$

where Γ is the customary Gamma function to be described below. Observe that by using (1.3) our computations became significantly less cumbersome. In order to see how (1.3) can lead us to a definition of $p^{\star\alpha}$ for all $\alpha > 0$, we introduce the following. Consider the Gamma function

$$\Gamma^+(x) := \int_0^\infty e^{-t}t^{x-1}dt \qquad\qquad (\mathrm{Re}(x) > 0).$$

By integration by parts, it follows that

$$\Gamma^+(x) = x^{-1}\int_0^\infty e^{-t}t^{x}dt = x^{-1}\Gamma^+(x+1).$$

Thus, $x\Gamma^+(x) = \Gamma^+(x+1)$, or more generally for all $n \in \mathbb{N},\ \mathrm{Re}(x) > 0$,

$$\Gamma^+(x+n) = (x+n-1)(x+n-2)\cdots(x)\Gamma^+(x) \tag{1.8}$$

and

$$\lim_{x\to 0^+} x\Gamma(x) = \lim_{x\to 0}\Gamma^+(x+1) = 1.$$

Now if $x > 0$, $x \notin \{-1,-2,-3,...\}$, we define

$$\Gamma^-(x) = \frac{\Gamma^+(x+n)}{x(x+1)(x+n-1)}$$

where n is such that $n + x > 0$. Notice that the definition is independent of the choice for n; i.e., if $x + n > 0$ and $x + m > 0$ for some $m, n \in \mathbb{N}$ with $n < m$ then there exists $j \in \mathbb{N}$ such that $m = n + j$. Thus, by (1.8),

$$\begin{aligned} & \frac{\Gamma^+(x+n+j)}{x(x+1)\cdots(x+n-1)(x+n)\cdots(x+n+j-1)} \qquad (1.9)\\ = \ & \frac{(x+n+j-1)\cdots(x+n)\Gamma^+(x+n)}{x(x+1)\cdots(x+n-1)(x+n)\cdots(x+n+j-1)}\\ = \ & \frac{\Gamma^+(x+n)}{x(x+1)\cdots(x+n-1)(x+n)\cdots(x+n+j-1)}\\ = \ & \frac{\Gamma^+(x+n)}{x(x+1)\cdots(x+n-1)}. \end{aligned}$$

In this way we may extend the Γ function to $\mathbb{R}\setminus\{0,-1,-2,3,...\}$ by defining

$$\Gamma(x) := \begin{cases} \Gamma^-(x) & \text{if } x \in (-\infty,0)\setminus\{-1,-2,-3,...\}, \\ \Gamma^+(x) & \text{if } x > 0. \end{cases} \tag{1.10}$$

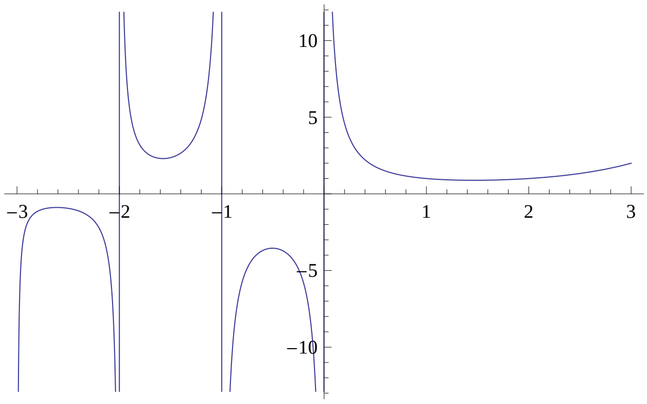


FIGURE 1.3. Graph of $\Gamma(x)$

It follows from the definition of $\Gamma(x)$ and (1.8) that

$$\Gamma(x+n) = (x+n-1)(x+n-2)\cdots(x)\Gamma(x) \tag{1.11}$$

for all $x \in \mathbb{R} \setminus \{0,-1,-2,3,...\}$ and all $n \in \mathbb{N}$. In particular,

$$\frac{\Gamma(x+n)}{\Gamma(x)} = (x+n-1)(x+n-2)\cdots(x+1)(x) \tag{1.12}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, where we take the right hand side of (1.12) as the definition of $\frac{\Gamma(x+n)}{\Gamma(x)}$ if $x \in \{0,-1,-2,-3,...\}$ and where we set $\frac{\Gamma(x+n)}{\Gamma(x)} = 1$ if $n = 0$.

Moreover, since

$$\binom{n+1}{j} = \frac{(n+j)!}{j!n!} = \frac{\Gamma(j+n+1)}{\Gamma(j+1)\Gamma(n+1)},$$

we define

$$\Psi_j^b := \binom{b+j}{j} := \frac{\Gamma(b+1+j)}{\Gamma(j+1)\Gamma(b+1)} = \frac{1}{j!}\frac{\Gamma(b+1+j)}{\Gamma(b+1)},$$

and we observe that Ψ_j^b is well-defined for all $b \in \mathbb{R}$ and $j \in \mathbb{N}_0$. In particular, by (1.12),

$$\begin{aligned} \Psi_j^{-\alpha-1} &= \frac{\Gamma(j+n+1)}{\Gamma(j+1)\Gamma(n+1)} = \frac{1}{j!}(-\alpha)(-\alpha-1)\cdots(\alpha-(j-1)) \\ &= \frac{1}{j!}(-1)^j\alpha(\alpha-1)\cdots(\alpha-(j-1)) = (-1)^j\binom{\alpha}{j}, \end{aligned} \tag{1.13}$$

where $\binom{\alpha}{j} := \frac{\alpha(\alpha-1)\cdots(\alpha-(j-1))}{j!}$ for all $\alpha \in \mathbb{R}, j \in \mathbb{N}_0$.

Observe that $\Psi_0^{-\alpha-1} = 1$ for all $\alpha \in \mathbb{R}$ and that

$$\Psi_j^{-\alpha-1} = 0 \tag{1.14}$$

if $\alpha \in \{0, -1, -2, ...\}$ and $j - \alpha > 0$. This allows us to rewrite $p^{\star n}(u)$ in (1.6) as

$$\begin{aligned} p^{\star n}(u) &= \Gamma(n) \sum_{j=0}^{[u]} \binom{-n-1+j}{j} (u-j)^{n-1} \\ &= \frac{1}{\Gamma(n)} \sum_{j=0}^{[u]} \Psi_j^{-n-1} (u-j)^{n-1}. \end{aligned}$$

We may "guess" that

$$p_\alpha(u) := \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{[u]} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1} \tag{1.15}$$

constitutes the "natural" candidate for p_α. In the next section we will see that this is indeed the case; i.e., we will show that p_α as defined in (1.15) is indeed a convolution semigroup $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$ with $p_1(u) = p(u) = \chi_{[0,1]}(u)$.

1.2 The Convolution Semigroup Generated by the Characteristic Function $u \mapsto \chi_{[0,1]}(u)$

In 1974, Ursula Westphal [30] showed that the characteristic function $p(u) = \chi_{[0,1]}(u)$ can be embedded into a convolution semigroup; i.e., there exists a family $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$ with $p_1 = p$, and $p_{\alpha+\beta}(u) = p_\alpha \star p_\beta$ for all $\alpha, \beta > 0$. What follows is nearly an identical reproduction of her result, with some added observations that will be of importance later in this dissertation. We begin by stating the following technical lemma.

Lemma 1.1. *Let Ψ_j^b be defined as in (1.13). Then*

(i) $\Psi_j^b = O(j^b)$ *as* $j \to \infty$*; i.e., there exists* $M > 0$ *such that* $|\Psi_j^b| \le M j^b$ *for all* $j \in \mathbb{N}$.

(ii) $\displaystyle\sum_{j=0}^{k} \Psi_j^b \Psi_{k-j}^\gamma = \Psi_k^{b+\gamma+1}$.

(iii) (Binomial Formula) Let $\alpha > 0$, then

$$(1-x)^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} x^j = \sum_{j=0}^{\infty} \Psi_j^{-\alpha-1} x^j \tag{1.16}$$

if $|x| < 1$.

(iv) Let $m \in \mathbb{N}$, $\kappa \in \mathbb{R} \setminus \{-1, -2, -3, ...\}$, and $0 < \theta \leq 1$. Then, for every $h \in \mathbb{N}_0$ there exists q_i and d_κ such that

$$(m+\theta)^\kappa = \sum_{i=0}^{h} q_i(\theta) \Psi_m^{\kappa-i} + d_\kappa(m, \theta), \tag{1.17}$$

where $\theta \mapsto q_i(\theta)$ are polynomials of degree $0 \leq i \leq h$ in θ with coefficients depending only on κ with $q_0(\theta) = \Gamma(\kappa+1)$ and $d_\kappa(m,\theta) = O(m^{\kappa-h-1})$ as $m \to \infty$.

Proof of (i). See [31, p. 390]. □

Proof of (ii). See [30]. □

Proof of (iii). Let $f(x) = (1-x)^\alpha = e^{\alpha \ln(1-x)}$ for $|x| < 1$. Then $\frac{f^{(j)}(0)}{j!} = \frac{(-1)^j}{j!}\alpha(\alpha - 1)\cdots(\alpha-(j-1)) = \Psi_j^{-\alpha-1}$. Therefore, by Taylor's theorem

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j = \sum_{j=0}^{\infty} \Psi_j^{-\alpha-1} x^j.$$

□

Proof of (iv). See Lemma 2.1 of [17]. □

Proposition 1.2. *For $\alpha > 0$ let*

$$p_\alpha(u) := \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{[u]} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1} \qquad (0 < u < \infty),$$

where $[u]$ denotes the integer part of u. Then

(i) for all $\alpha > 0$, $p_\alpha(u) \in L^1(0,\infty)$, $\int_0^\infty p_\alpha(u)\,du = 1$, and $\widehat{p_\alpha}(z) = \left(\frac{1-e^{-z}}{z}\right)^\alpha$ for $Re(z) > 0$.

(ii) $p_1 = \chi_{[0,1]}$, *and* $p_{\alpha+\beta} = p_\alpha \star p_\beta$ *for all* $\alpha, \beta > 0$.

(iii) $p_n \geq 0$ *for all* $n \in \mathbb{N}$ *and* $supp[p_n] = [0, n]$. *In particular, for* $n \geq 2$, p_n *is continuous and, for* $n \geq 3$, p_n *is* $(n-2)$*-times continuously differentiable on* $\mathbb{R}$.

(iv) For $\alpha \in (0,1)$, $u \mapsto p_\alpha(u)$ *is discontinuous at* $u = n \in \mathbb{N}$. *More precisely* $\lim_{u \to n^+} p_\alpha(u) = -\infty$, *and* $\lim_{u \to n^-} p_\alpha(u)$ *exists. In particular, if* $\alpha \in (0,1)$, *then* p_α *is not a strictly positive function.*

(v) For $\alpha > 1$, p_α *is continuous on* $(0, \infty)$ *with* $p_\alpha(0) = 0$.

Proof of (i). Let $\alpha > 0$. To show that $\int_0^{n+1} |p_\alpha| du$ exists, observe that

$$\begin{aligned}\int_0^{n+1} |p_\alpha(u)| du &= \sum_{k=0}^{n} \int_k^{k+1} |p_\alpha(u)| du \\ &= \sum_{k=0}^{n} \int_k^{k+1} |\frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1}| du.\end{aligned}$$

For $0 \leq j \leq k-1$, the functions $u \mapsto \Psi_j^{-\alpha-1}(u-j)^{\alpha-1}$ are continuous for $u \in [k, k+1]$, and for $j = k$ the function $u \mapsto \Psi_j^{-\alpha-1}(u-k)^{\alpha-1}$ is in $L^1(k, k+1)$ since $\alpha > 0$. Thus $\displaystyle\int_0^{n+1} |p_\alpha(u)| du$ exists.

To discuss the existence of $\displaystyle\int_{n+1}^{\infty} |p_\alpha(u)|\, du$, observe that

$$\int_{n+1}^{\infty} |p_\alpha(u)| du = \sum_{k=n+1}^{\infty} \int_k^{k+1} |p_\alpha(u)| du \tag{1.18}$$

$$= \sum_{k=n+1}^{\infty} \int_0^1 |p_\alpha(k+\theta)| d\theta. \tag{1.19}$$

We now apply Lemma 1.1 for $\kappa = \alpha - 1$, $h = n$, and $m = k - j$ to each of the terms of p_α as follows.

$$
\begin{aligned}
p_\alpha(k+\theta) &= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\Psi_j^{-\alpha-1}(k-j+\theta)^{\alpha-1} \\
&= \frac{1}{\Gamma(\alpha)}\Psi_k^{-\alpha-1}\theta^\alpha + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\Psi_j^{-\alpha-1}\sum_{i=0}^{n} q_i(\theta)\Psi_{k-j}^{\alpha-i-1} \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\Psi_j^{-\alpha-1}d_{\alpha-1}(k-j,\theta). \qquad (1.20)
\end{aligned}
$$

We now treat each term of (1.20) separately. Let

$$I := \frac{1}{\Gamma(\alpha)}\Psi_k^{-\alpha-1}\theta^\alpha,$$

$$II := \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\Psi_j^{-\alpha-1}\sum_{i=0}^{n} q_i(\theta)\Psi_{k-j}^{\alpha-i-1}, \text{ and}$$

$$III := \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\Psi_j^{-\alpha-1}d_{\alpha-1}(k-j,\theta).$$

Since $0 \le \theta \le 1$, it follows from (i) of Lemma 1.1 that

$$I = \Psi_k^{-\alpha-1}\theta^{\alpha-1} = O(k^{-\alpha-1}). \qquad (1.21)$$

The term II may be estimated by noticing that $k \ge n+1$ and by applying Lemma 1.1 as follows

$$
\begin{aligned}
II &= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\Psi_j^{-\alpha-1}\sum_{i=0}^{n} q_i(\theta)\Psi_{k-j}^{\alpha-i-1} \qquad (1.22) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n} q_i(\theta)\sum_{j=0}^{k-1}\Psi_j^{-\alpha-1}\Psi_{k-j}^{\alpha-i-1} \\
&= \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n} q_i(\theta)[\Psi_k^{-i-1} - \Psi_k^{-\alpha-1}\Psi_0^{\alpha-i-1}] \\
&= \frac{-\Psi_k^{-\alpha-1}}{\Gamma(\alpha)}\sum_{i=0}^{n} q_i(\theta) = O(k^{-\alpha-1})
\end{aligned}
$$

since, by (1.13),

$$\Psi_k^{-i-1} = \frac{(-1)^k}{k!}i(i-1)\cdots(i-(k-1)) = 0$$

for all $0 \leq i \leq n < k$.

For the estimation of III, we first observe that for any fixed $\alpha > 0$, $\Psi_0^{-\alpha-1} = 1$ (see (1.13). Therefore, by Lemma 1.1, we have

$$\begin{aligned}
\sum_{j=0}^{k-1} \Psi_j^{-\alpha-1} d(k-j,\theta) &= d_{\alpha-1}(k,\theta) + \sum_{j=1}^{k-1} \Psi_j^{-\alpha-1} d_{\alpha-1}(k-j,\theta) \\
&= O(k^{\alpha-n-2}) + \sum_{j=1}^{k-1} O(j^{-\alpha-1}) O((k-j)^{\alpha-n-2}) \\
&= O(k^{\alpha-n-2}) + O(k^{\alpha-n-2}) \sum_{j=1}^{[k/2]} O(j^{-\alpha-1}) \\
&\qquad + O(k^{-\alpha-1}) \sum_{j=[k/2]+1}^{k-1} O((k-j)^{\alpha-n-2}) \\
&= O(k^{\alpha-n-2} + k^{-\alpha-1}). \qquad (1.23)
\end{aligned}$$

Finally, it follows from (1.20) and the estimates we obtained for I, II, and III that

$$p_\alpha(k+\theta) = I + II + III = O(k^{-\alpha-1}) + O(k^{\alpha-n-2} + k^{-\alpha-1}).$$

Therefore,

$$\int_{n+1}^{\infty} |p_\alpha(u)| du = \sum_{k=n+1}^{\infty} O(k^{\alpha-n-2} + k^{-\alpha-1}) < \infty.$$

It follows that $p_\alpha \in L^1(0,\infty)$ for all $\alpha > 0$ and that

$$\widehat{p_\alpha}(z) := \int_0^\infty e^{-zu} p(u)\, du$$

exists for all $\mathrm{Re}(z) \geq 0$.

On the other hand, from the Binomial Formula ((iii) of Lemma 1.1), we obtain that

$$(1-e^{-z})^\alpha = \sum_{j=0}^{\infty} \Psi_j^{-\alpha-1} e^{-jz}. \qquad (1.24)$$

Now by (1.24) and Fubini's theorem, it follows that

$$
\begin{aligned}
\int_0^\infty e^{-zu}p_\alpha(u)du &= \int_0^\infty e^{-zu}\left(\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{[u]}\Psi_j^{-\alpha-1}(u-j)^{\alpha-1}\right)du\\
&= \sum_{k=0}^\infty \int_k^{k+1} e^{-zu}\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{[u]}\Psi_j^{-\alpha-1}(u-j)^{\alpha-1}du\\
&= \sum_{k=0}^\infty \int_k^{k+1} e^{-zu}\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\Psi_j^{-\alpha-1}(u-j)^{\alpha-1}du\\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty \int_k^{k+1}\sum_{j=0}^{k} e^{-zu}\Psi_j^{-\alpha-1}(u-j)^{\alpha-1}du\\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty\sum_{j=0}^{k} \int_k^{k+1} e^{-zu}\Psi_j^{-\alpha-1}(u-j)^{\alpha-1}du\\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty\sum_{j=0}^{k} \int_{k-j}^{k-j+1} e^{-zj}e^{-zw}\Psi_j^{-\alpha-1}w^{\alpha-1}dw\\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^\infty\sum_{j=0}^{k} e^{-zj}\Psi_j^{-\alpha-1}\int_{k-j}^{k-j+1} e^{-zw}w^{\alpha-1}dw\\
&= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^\infty\sum_{k=j}^{\infty} e^{-zj}\Psi_j^{-\alpha-1}\int_{k-j}^{k-j+1} e^{-zw}w^{\alpha-1}dw\\
&= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^\infty e^{-zj}\Psi_j^{-\alpha-1}\sum_{k=j}^{\infty}\int_{k-j}^{k-j+1} e^{-zw}w^{\alpha-1}dw\\
&= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^\infty e^{-zj}\Psi_j^{-\alpha-1}\int_0^\infty e^{-zw}w^{\alpha-1}dw\\
&= \sum_{j=0}^\infty e^{-zj}\Psi_j^{-\alpha-1}\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-zw}w^{\alpha-1}dw\\
&= \sum_{j=0}^\infty e^{-zj}\Psi_j^{-\alpha-1}\frac{1}{z^\alpha} = \left(\frac{1-e^{-z}}{z}\right)^\alpha.
\end{aligned}
$$

□

By the dominated convergence theorem $\left(\frac{1-e^{-z}}{z}\right)^\alpha = \int_0^\infty e^{-zu}p_\alpha(u)du \to \int_0^\infty p_\alpha(t)dt$ as $z \to 0$. Since $\left(\frac{1-e^{-z}}{z}\right)^\alpha \to 1$ as $z \to 0$, it follows that $\int_0^\infty p_\alpha(t)dt = 1$ for all $\alpha > 0$.

Proof of (ii). Since,

$$\widehat{p_1}(z) = \frac{1-e^{-z}}{z} = \int_0^\infty e^{-zu}\chi_{[0,1]}(u)\,du,$$

the uniqueness of the Laplace transform (see [1, p. 41]) implies that $p_1(u) = \chi_{[0,1]}(u)$. Similarly, since

$$\begin{aligned}
\widehat{p_{\alpha+\beta}}(z) &= \left(\frac{1-e^{-z}}{z}\right)^{\alpha+\beta} = \left(\frac{1-e^{-z}}{z}\right)^{\alpha}\left(\frac{1-e^{-z}}{z}\right)^{\beta} \\
&= \widehat{p_\alpha}(z)\widehat{p_\beta}(z) = \widehat{p_\alpha \star p_\beta}(z),
\end{aligned}$$

it follows that $p_{\alpha+\beta} = p_\alpha \star p_\beta$ as sought. □

Proof of (iii). Let $0 \le f$ be a continuous function with $supp[f] = [0,n]$, and let $g \in L^1(0,\infty)$ with $supp[g] = [0,1]$. Then

$$(f \star g)(u) = \begin{cases} \int_0^u f(u-s)g(s)\,ds & \text{if } 0 \le u \le 1, \\ \int_0^1 f(u-s)g(s)\,ds & \text{if } u > 1. \end{cases}$$

Now if $u \ge n+1$, then $u - s \ge n$ for all $s \in [0,1]$, and thus $f(u-s) = 0$ for all $s \in [0,1]$. It follows that $(f \star g)(u) = 0$ for all $u \ge n+1$; i.e., $supp[f \star g] \subset [0, n+1]$. Moreover, if $f(u) > 0$ for $u \in (0,n)$ and $g(u) > 0$ for $u \in (0,1)$, then $supp[f \star g] = [0, n+1]$, since the integrals that define $(f \star g)$ can never be zero. Therefore, by an inductive argument one verifies that $supp[p^{\star n}] = [0,n]$. We also note, to conclude the proof of (iii), that the convolution of two positive functions must be a positive function. Therefore, $p_n \ge 0$ for all $n \in \mathbb{N}$. Now let f be $(n-2)$-times continuously differentiable on $\mathbb{R}$ with $supp[f] = [0,n]$. Then

$$(f \star p)(u) = \begin{cases} \int_0^u f(u-s)\,ds & \text{if } 0 \le u \le 1, \\ \int_0^1 f(u-s)\,ds & \text{if } u > 1 \end{cases} = \begin{cases} \int_0^u f(s)\,ds & \text{if } 0 \le u \le 1, \\ \int_{u-1}^u f(u-s)\,ds & \text{if } u > 1, \end{cases}$$

shows that $(f \star p)$ is $(n-1)$-times continuously differentiable. □

Proof of (iv). If $0 < \alpha < 1$, then

$$p_\alpha(u) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{[u]} \Psi_j^{-\alpha-1} \frac{1}{(u-j)^{1-\alpha}}. \tag{1.25}$$

Therefore, the only possible singularities of (1.25) are those where $\frac{1}{(u-j)^{1-\alpha}}$ is undefined. These points are clearly the positive integers. Now let $m \in \mathbb{N}$. Observe first that $\Psi_m^{-\alpha-1} = (-1)^m = \alpha(\alpha-1)\cdots(\alpha-(m-1)) < 0$ if $0 < \alpha < 1$. Then

$$\begin{aligned}
\lim_{u\to m^+} p_\alpha(u) &= \lim_{u\to m^+} \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{m} \frac{\Psi_j^{-\alpha-1}}{(u-j)^{1-\alpha}} \\
&= \lim_{u\to m^+} \frac{1}{\Gamma(\alpha)} \left\{ \sum_{j=0}^{m-1} \frac{\Psi_j^{-\alpha-1}}{(u-j)^{1-\alpha}} + \frac{\Psi_m^{-\alpha-1}}{(u-m)^{1-\alpha}} \right\} \\
&= \frac{1}{\Gamma(\alpha)} \left\{ \sum_{j=0}^{m-1} \frac{\Psi_j^{-\alpha-1}}{(m-j)^{1-\alpha}} + \lim_{u\to m^+} \frac{\Psi_m^{-\alpha-1}}{(u-m)^{1-\alpha}} \right\} = -\infty
\end{aligned}$$

and

$$\begin{aligned}
\lim_{u\to m^-} p_\alpha(u) &= \lim_{u\to m^-} \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{m-1} \frac{\Psi_j^{-\alpha-1}}{(m-j)^{1-\alpha}} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{m-1} \frac{\Psi_j^{-\alpha-1}}{(m-j)^{1-\alpha}} < \infty.
\end{aligned}$$

□

Proof of (v). Let $\alpha > 1$ and $u \in [n, n+1)$. Then

$$p_\alpha(u) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1}$$

is obviously continuous for $u \in (n, n+1)$, and

$$p_\alpha(n) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \Psi_j^{-\alpha-1} (n-j)^{\alpha-1} = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1}.$$

If $u \in [n-1, n)$, then

$$p_\alpha(u) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1}$$

and therefore

$$\begin{aligned}
\lim_{u \to n^-} p_\alpha(u) &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \Psi_j^{-\alpha-1} (n-j)^{\alpha-1} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \Psi_j^{-\alpha-1} (n-j)^{\alpha-1} \\
&= p_\alpha(n).
\end{aligned}$$

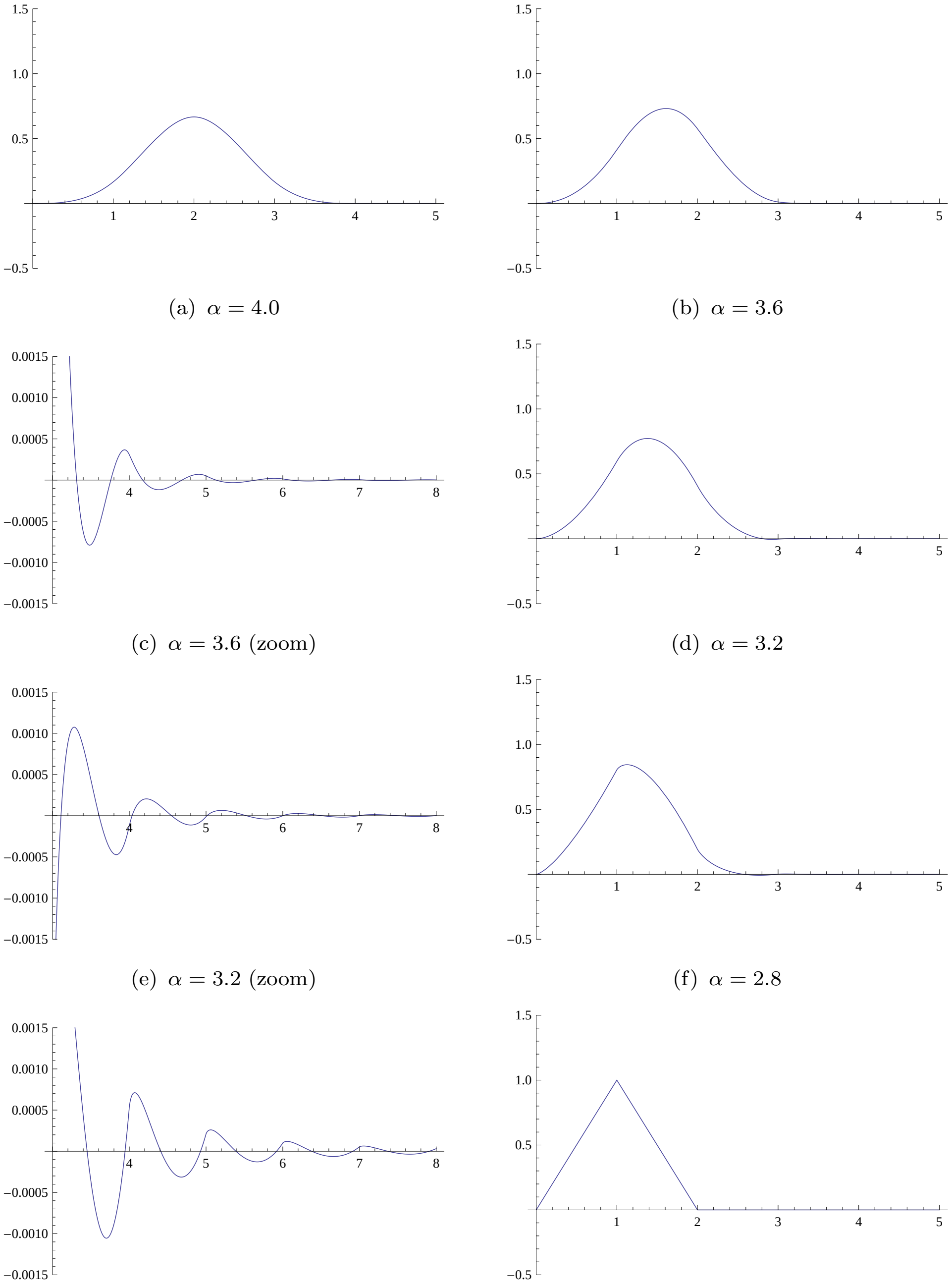
This shows that $p_\alpha(n)$ is continuous on $(0, \infty)$. Moreover, since $\alpha > 0$, then it clearly follows that

$$\begin{aligned}
\lim_{u \to 0^+} p_\alpha(u) &= \lim_{u \to 0^+} \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{[u]} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1} \\
&= \lim_{u \to 0^+} \frac{1}{\Gamma(\alpha)} \Psi_0^{-\alpha-1} u^{\alpha-1} \\
&= \frac{1}{\Gamma(\alpha)} \lim_{u \to 0^+} u^{\alpha-1} = 0.
\end{aligned}$$

Thus the proof of (v) is complete. $\square$

In the following figures we illustrate the results of Proposition 1.2. These graphs are obtained by plotting the formula for $p^{\star\alpha}$ given by the summation; i.e.,

$$p_\alpha(u) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{[u]} \Psi_j^{-\alpha-1} (u-j)^{\alpha-1}.$$



(a) $\alpha = 4.0$

(b) $\alpha = 3.6$

(c) $\alpha = 3.6$ (zoom)

(d) $\alpha = 3.2$

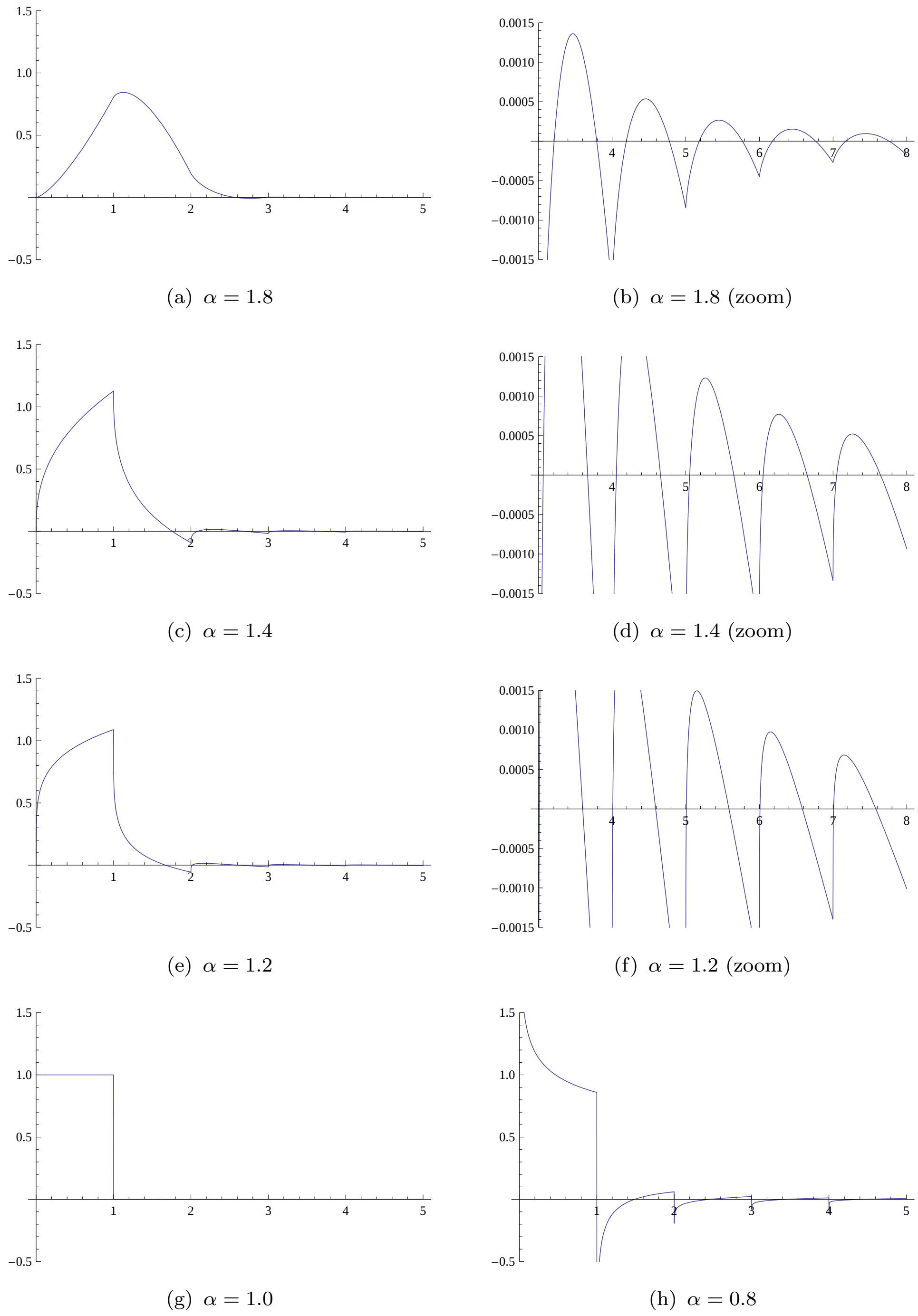
(e) $\alpha = 3.2$ (zoom)

(f) $\alpha = 2.8$

(g) $\alpha = 2.8$ (zoom)

(h) $\alpha = 2.0$

FIGURE 1.4. $\chi_{[0,1]}^{\star\alpha}(u)$



(a) $\alpha = 1.8$

(b) $\alpha = 1.8$ (zoom)

(c) $\alpha = 1.4$

(d) $\alpha = 1.4$ (zoom)

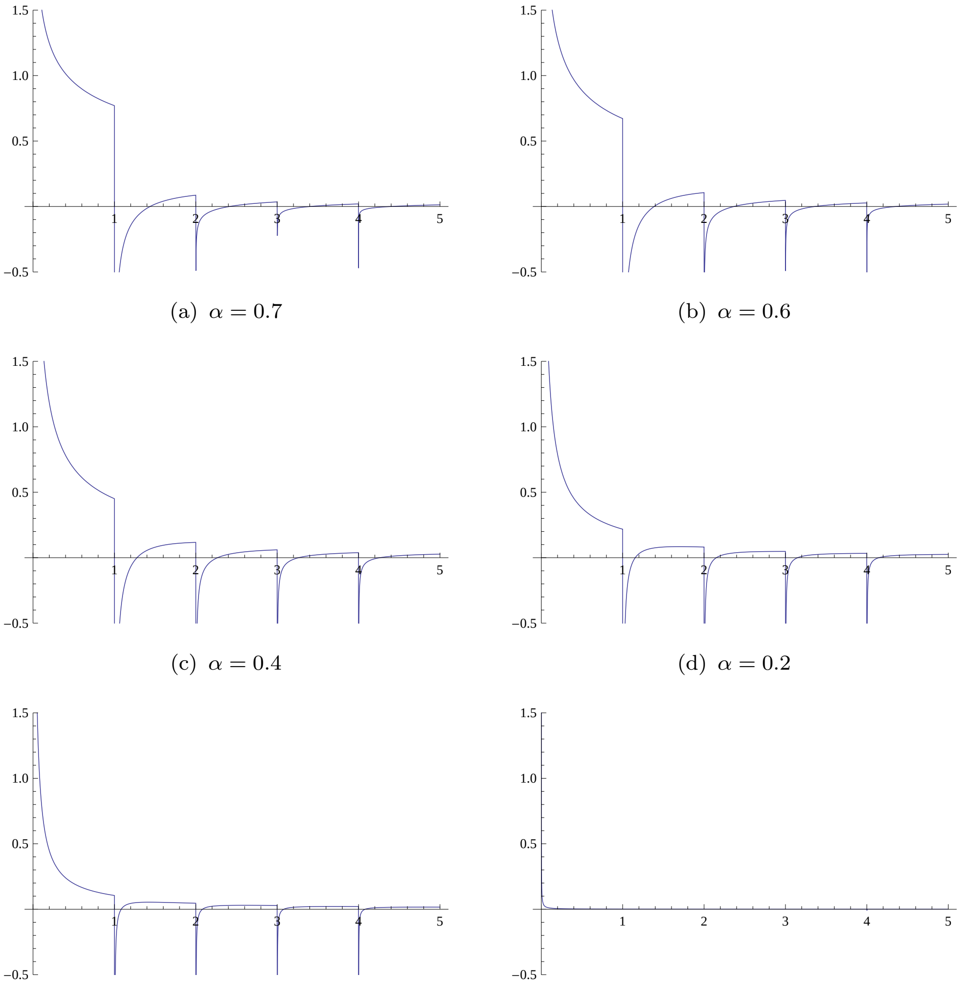
(e) $\alpha = 1.2$

(f) $\alpha = 1.2$ (zoom)

(g) $\alpha = 1.0$

(h) $\alpha = 0.8$

FIGURE 1.5. $\chi_{[0,1]}^{\star\alpha}(u)$



(a) $\alpha = 0.7$ (b) $\alpha = 0.6$

(c) $\alpha = 0.4$ (d) $\alpha = 0.2$

(e) $\alpha = 0.1$ (f) $\alpha = 0.01$

FIGURE 1.6. $\chi_{[0,1]}^{\star\alpha}$

1.3 Convolution Semigroups Generated by Step Functions

After investigating the convolution semigroup generated the characteristic function on the interval $[0, 1]$, we now apply the same methodology to a slightly more general class of functions. The class of step functions are still simple enough so that the computations similar to those found in the previous chapter can be adapted, but general enough to gain insights into the underlying problems and challenges. Let $\tau > 0$. As a first step, we will consider the function

$$g(u) := \frac{1}{\tau} p(u/\tau) = \frac{1}{\tau} \chi_{[0,1]}(u/\tau) = \frac{1}{\tau} \chi_{[0,\tau]}(u)$$

whose Laplace transform is given by

$$\widehat{g}(z) = \frac{1 - e^{-\tau z}}{\tau z}$$

for $z \neq 0$. Define

$$g_\alpha(u) := \frac{1}{\tau} p_\alpha(u/\tau) := \frac{1}{\tau} \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{[u/\tau]} \Psi_j^{-\alpha-1} (u/\tau - j)^{\alpha-1}, \tag{1.26}$$

where p_α is defined as in Proposition 1.2. Then, for $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$,

$$\begin{aligned} \widehat{g_\alpha}(z) &= \int_0^\infty e^{-zu} \frac{1}{\tau} p_\alpha(\frac{u}{\tau})\, du = \int_0^\infty e^{-z\tau w} p_\alpha(w)\, dw \\ &= \widehat{p_\alpha}(\tau w) = (\frac{1 - e^{-\tau z}}{\tau z})^\alpha. \end{aligned} \tag{1.27}$$

By the uniqueness of the Laplace transform, $\widehat{g_1} = \widehat{g}$ and $\widehat{g_\alpha \star g_\beta} = \widehat{g_\alpha} \cdot \widehat{g_\beta} = \widehat{g_{\alpha+\beta}}$ $(\alpha, \beta > 0)$ implies that $g_1 = g$ and $g_\alpha \star g_\beta = g_{\alpha+\beta}$. This shows that the convolution semigroup $(g_\alpha)_{\alpha>0}$ generated by the characteristic function of the interval $[0, \tau]$ is given by (1.27) (see also Proposition 1.2 below.)

In order to determine the convolution semigroups generated by more general step functions we need an intermediate step.

Lemma 1.3. *(a) Let $\alpha \geq \beta > 0$, $\tau > 0$ and consider*

$$q_{\alpha,\beta,\tau}(u) := \frac{\tau}{\Gamma(\beta)} \sum_{j=0}^{[\tau u]} \psi_j^{-\alpha-1} (\tau u - j)^{\beta-1} \quad (u > 0).$$

Then $q_{\alpha,\beta,\tau} \in L^1(0,\infty)$ and $\widehat{q_{\alpha,\beta,\tau}} = \tau^\beta \frac{(1-e^{\frac{-z}{\tau}})^\alpha}{z^\beta}$.
(b) Let $0 < \beta \leq \alpha$, $|c| \leq 1$, $\tau > 0$, and consider

$$h_{\alpha,\beta,\tau,c}(u) := \frac{\tau}{\Gamma(\beta)} \sum_{j=0}^{[u\tau]} (-1)^j \psi_j^{-\alpha-1} c^j (u\tau - j)^{\beta-1}$$

Then $h_{\alpha,\beta,\tau,c} \in L^1(0,\infty)$ and

$$\widehat{h_{\alpha,\beta,\tau,c}}(z) = \frac{\tau^\beta (1 + ce^{\frac{-z}{\tau}})^\alpha}{z^\beta} \quad (Re(z) > 0)$$

Proof. The proofs that $q_{\alpha,\beta,\tau} \in L^1(0,\infty)$ and $h_{\alpha,\beta,\tau,c} \in L^1(0,\infty)$ proceed along the same lines as the proof of $p_\alpha \in L^1(0,\infty)$ in Proposition 1.2. Moreover, it is sufficient to prove (a) and (b) for the case $\tau = 1$. The general case follows then immediately from the fact that if $\widehat{f}$ is the Laplace transform of f, then $z \to \widehat{f}(\frac{z}{\tau})$ is the Laplace transform of $u \to \tau f(u\tau)$. Let $\mathrm{Re}(z) > 0$, then □

$$
\begin{aligned}
\widehat{q_{\alpha,\beta,1}}(z) &= \int_0^\infty e^{-zu} q_{\alpha,\beta,1}(u)du \\
&= \int_0^\infty e^{-zu} \left(\frac{1}{\Gamma(\beta)} \sum_{j=0}^{[u]} \Psi_j^{-\alpha-1} (u-j)^{\beta-1} \right) du \\
&= \sum_{k=0}^\infty \int_k^{k+1} e^{-zu} \frac{1}{\Gamma(\beta)} \sum_{j=0}^{[u]} \Psi_j^{-\alpha-1} (u-j)^{\beta-1} du \\
&= \sum_{k=0}^\infty \int_k^{k+1} e^{-zu} \frac{1}{\Gamma(\beta)} \sum_{j=0}^{k} \Psi_j^{-\alpha-1} (u-j)^{\beta-1} du \\
&= \frac{1}{\Gamma(\beta)} \sum_{k=0}^\infty \int_k^{k+1} \sum_{j=0}^{k} e^{-zu} \Psi_j^{-\alpha-1} (u-j)^{\beta-1} du \\
&= \frac{1}{\Gamma(\beta)} \sum_{k=0}^\infty \sum_{j=0}^{k} \int_k^{k+1} e^{-zu} \Psi_j^{-\alpha-1} (u-j)^{\beta-1} du \\
&= \frac{1}{\Gamma(\beta)} \sum_{k=0}^\infty \sum_{j=0}^{k} \int_{k-j}^{k-j+1} e^{-zj} e^{-zw} \Psi_j^{-\alpha-1} w^{\beta-1} dw \\
&= \frac{1}{\Gamma(\beta)} \sum_{k=0}^\infty \sum_{j=0}^{k} e^{-zj} \Psi_j^{-\alpha-1} \int_{k-j}^{k-j+1} e^{-zw} w^{\beta-1} dw \\
&= \frac{1}{\Gamma(\beta)} \sum_{j=0}^\infty \sum_{k=j}^{\infty} e^{-zj} \Psi_j^{-\alpha-1} \int_{k-j}^{k-j+1} e^{-zw} w^{\beta-1} dw \\
&= \frac{1}{\Gamma(\beta)} \sum_{j=0}^\infty e^{-zj} \Psi_j^{-\alpha-1} \sum_{k=j}^{\infty} \int_{k-j}^{k-j+1} e^{-zw} w^{\beta-1} dw \\
&= \frac{1}{\Gamma(\beta)} \sum_{j=0}^\infty e^{-zj} \Psi_j^{-\alpha-1} \int_0^\infty e^{-zw} w^{\beta-1} dw \\
&= \sum_{j=0}^\infty e^{-zj} \Psi_j^{-\alpha-1} \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-zw} w^{\beta-1} dw \\
&= \sum_{j=0}^\infty e^{-zj} \Psi_j^{-\alpha-1} \frac{1}{z^\beta} = \frac{(1-e^{-z})^\alpha}{z^\beta}.
\end{aligned}
$$

We note here that the last equality is valid due to the Binomial Formula (1.16).

Similarly,

$$\begin{aligned}
\widehat{h_{\alpha,\beta,c,1}}(z) &= \int_0^\infty e^{-zu}\left(\frac{1}{\Gamma(\beta)}\sum_{j=0}^{[u]}\Psi_j^{-\alpha-1}(-1)^j c^j (u-j)^{\beta-1}\right) \\
&= \sum_{k=0}^{\infty}\int_k^{k+1} e^{-zu}\frac{1}{\Gamma(\beta)}\sum_{j=0}^{[u]}\Psi_j^{-\alpha-1}(-1)^j c^j (u-j)^{\beta-1}du \\
&= \sum_{k=0}^{\infty}\int_k^{k+1} e^{-zu}\frac{1}{\Gamma(\beta)}\sum_{j=0}^{k}\Psi_j^{-\alpha-1}(-1)^j c^j (u-j)^{\beta-1}du \\
&= \frac{1}{\Gamma(\beta)}\sum_{k=0}^{\infty}\int_k^{k+1}\sum_{j=0}^{k} e^{-zu}\Psi_j^{-\alpha-1}(-1)^j c^j (u-j)^{\beta-1}du \\
&= \frac{1}{\Gamma(\beta)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\int_k^{k+1} e^{-zu}\Psi_j^{-\alpha-1}(-1)^j c^j (u-j)^{\beta-1}du \\
&= \frac{c^\alpha}{\Gamma(\beta)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\int_k^{k+1} e^{-zj-zw}\Psi_j^{-\alpha-1}(-1)^j c^j w^{\beta-1}dw \\
&= \frac{1}{\Gamma(\beta)}\sum_{k=0}^{\infty}\sum_{j=0}^{k} e^{-zj}(-1)^j c^j\Psi_j^{-\alpha-1}\int_{k-j}^{k+j-1} e^{-zw}w^{\beta-1}dw \\
&= \frac{1}{\Gamma(\beta)}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}(-1)^j c^j e^{-zj}\Psi_j^{-\alpha-1}\int_{k-j}^{k+j-1} e^{-zw}w^{\beta-1}dw \\
&= \frac{1}{\Gamma(\beta)}\sum_{j=0}^{\infty}(-1)^j c^j e^{-zj}\Psi_j^{-\alpha-1}\sum_{k=j}^{\infty}\int_{k-j}^{k+j-1} e^{-zw}w^{\beta-1}dw \\
&= \frac{1}{\Gamma(\beta)}\sum_{j=0}^{\infty}(-1)^j c^j e^{-zj}\Psi_j^{-\alpha-1}\int_0^\infty e^{-zw}w^{\beta-1}dw \\
&= \sum_{j=0}^{\infty}(-1)^j c^j e^{-zj}\Psi_j^{-\alpha-1}\frac{1}{\Gamma(\beta)}\int_0^\infty e^{-zw}w^{\beta-1}dw \\
&= \sum_{j=0}^{\infty}(-1)^j c^j e^{-zj}\Psi_j^{-\alpha-1}\frac{1}{z^\beta} = \frac{1}{z^\beta}\sum_{j=0}^{\infty}(-1)^j c^j e^{-zj}\Psi_j^{-\alpha-1} = \frac{(1+ce^{-z})^\alpha}{z^\beta}.
\end{aligned}$$

Remark 1.4. For the preceding equalities to remain valid it is imperative that c is *chosen* such that $|c| \leq 1$ for the Binomial Formula (1.16) to hold.

With the remark above one sees that the function $z \mapsto \frac{\tau^\beta(c+e^{-z/\tau})^\alpha}{z^\beta}$ is the Laplace transform of

$$h_{\alpha,\beta,\tau,c}(u) = \frac{\tau}{\Gamma(\beta)} \sum_{j=0}^{[u\tau]} \Psi_j^{-\alpha-1}(-1)^j c^j (u\tau - j)^{\beta-1}. \tag{1.28}$$

We now discuss the convolution semigroup generated by a two-step function

$$q(u) = a\chi_{[0,\frac{1}{2}]}(u) + b\chi_{[\frac{1}{2},1]}(u). \tag{1.29}$$

Then, for $z \in \bar{\mathbb{C}_+} := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$,

$$\begin{aligned}
\widehat{q}(z) &= \frac{a(1-e^{\frac{z}{2}})}{z} + \frac{b(e^{-\frac{z}{2}} - e^{-z})}{z} \\
&= \frac{a}{z}(1-e^{-\frac{z}{2}}) + \frac{b}{z}e^{-\frac{z}{2}}(1-e^{-\frac{z}{2}}) \\
&= \frac{1}{z}(1-e^{-\frac{z}{2}})(a+be^{-\frac{z}{2}}).
\end{aligned}$$

In order to define $(a + be^{-\frac{z}{2}})^\alpha$ for some $\alpha > 0$ and all $z > 0$, we have to make sure that $a + be^{-\frac{z}{2}} > 0$ for all $z > 0$. This shows that a must be positive; i.e., we can always assume that $a \geq 0$.

Let us first treat the case $a = 0$ separately and consider a step function of the form

$$q(u) = \frac{1}{\tau - \epsilon}\chi_{[\epsilon,\tau]}(u)$$

for some $0 \leq \epsilon < \tau$. Then, for $z \neq 0$,

$$\widehat{q}(z) = \frac{1}{\tau-\epsilon}\frac{e^{-z\epsilon} - e^{-z\tau}}{z} = \frac{1}{\tau-\epsilon}\frac{e^{-\epsilon z}}{z}[1 - e^{-(\tau-\epsilon)z}]$$

and thus, for $0 < \beta < \alpha$ and $z > 0$,

$$\widehat{q}(z)^\alpha = e^{-\epsilon\alpha z}\left[\frac{1 - e^{-(\tau-\epsilon)z}}{z(\tau-\epsilon)}\right]^\alpha.$$

We showed in (1.28) that

$$g_{\alpha,\tau}(u) := \frac{1}{\tau}p_\alpha(u/\tau) := \frac{1}{\tau}\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{[u/\tau]} \Psi_j^{-\alpha-1}(u/\tau - j)^{\alpha-1} \tag{1.30}$$

is the convolution semigroup generated by the characteristic function $g_\tau(u) = \frac{1}{\tau}\chi_{[0,\tau]}(u)$, and that the Laplace transform is given by

$$\widehat{g_{\alpha,\tau}}(z) = \left(\frac{1-e^{-\tau z}}{\tau z}\right)^{\alpha}.$$

Since $z \mapsto e^{az}\widehat{f}(z)$ is the Laplace transform of $u \mapsto f(u-a)\chi_{[n,\infty)}(u)$, it follows that

$$(\widehat{q}(z))^{\alpha} = e^{-\epsilon\alpha z}\widehat{g_{\alpha,\tau-\epsilon}}(z)$$

is the Laplace transform of the function

$$\begin{aligned}
q_\alpha(u) &= g_{\alpha,\tau-\epsilon}(u-\epsilon\alpha)\chi_{[2\alpha,\infty)} \\
&= \frac{1}{\tau-\epsilon}\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{\left[\frac{u-\epsilon\alpha}{\tau-\epsilon}\right]}\Psi_j^{-\alpha-1}\left(\frac{u-\epsilon\alpha}{\tau-\epsilon}-j\right)^{\alpha-1} \\
&= \frac{1}{\tau-\epsilon}p_\alpha\left(\frac{u-\epsilon\alpha}{\tau-\epsilon}\right), \qquad (1.31)
\end{aligned}$$

where $\{p_\alpha\}_{\alpha>0}$ is the convolution semigroup generated by $p(u) = \chi_{[0,1]}(u)$.

Since

$$\widehat{q_\alpha \star q_\beta} = (\widehat{q})^{\alpha}(\widehat{q})^{\beta} = \widehat{q_{\alpha+\beta}},$$

it follows that $q_\alpha \star q_\beta = q_{\alpha+\beta}$. Moreover,

$$\begin{aligned}
q_1(u) &= \frac{1}{\tau-\epsilon}p_1\left(\frac{u-\epsilon}{\tau-\epsilon}\right) \\
&= \frac{1}{\tau-\epsilon}\chi_{[0,1]}\left(\frac{u-\epsilon}{\tau-\epsilon}\right) \\
&= \frac{1}{\tau-\epsilon}\chi_{[\epsilon,\tau]}(u),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty q_\alpha(u)\,du &= \frac{1}{\tau-\epsilon}\int_0^\infty p_\alpha\left(\frac{u-\epsilon\alpha}{\tau-\epsilon}\right)du \\
&= \int_{\frac{-\epsilon\alpha}{\tau-\epsilon}}^\infty p_\alpha(w)\,dw = \int_0^\infty p_\alpha(w)\,dw = 1.
\end{aligned}$$

Let us now go back to (1.29) and consider

$$g(u) = a\chi_{[0,\frac{1}{2}]}(u) + b\chi_{[\frac{1}{2},1]}(u)$$

for some $a > 0$. We showed that

$$\widehat{g}(z) = \frac{a}{z}(1 - e^{\frac{-z}{2}})(1 + \frac{b}{a}e^{-\frac{z}{2}})$$

for all $0 \neq z \in \mathbb{C}$. Thus, if the "jump size" condition $|b| \leq a$ holds, then

$$(\widehat{g}(z))^{\alpha} = \frac{a^{\alpha}}{z^{\alpha}}(1 - e^{\frac{-z}{2}})^{\alpha}(1 + \frac{b}{a}e^{-\frac{z}{2}})^{\alpha}$$

is well defined for all $z \in \mathbb{C}_+$ and, for all $0 < \beta < \alpha$,

$$(\widehat{g}(z))^{\alpha} = (\frac{a}{2})^{\alpha}\left[\frac{2^{\beta}(1 - e^{\frac{-z}{2}})^{\alpha}}{z^{\beta}}\right]\left[\frac{2^{\alpha-\beta}(1 + \frac{b}{a}e^{-\frac{z}{2}})^{\alpha}}{z^{\alpha-\beta}}\right].$$

Define

$$g_{\alpha}(u) := \left(\frac{a}{2}\right)^{\alpha} q_{\alpha,\beta,2} \star h_{\alpha,\alpha-\beta,2,\frac{b}{a}},$$

where $q_{\alpha,\beta,2}$ and $h_{\alpha,\beta,\tau,c}$ are as in Lemma 1.3. Then $g_{\alpha} \in L^1(0,\infty)$.

Since $\widehat{g_{\alpha}}(z) = (\widehat{g})^{\alpha}$, it follows that $\widehat{g_1}(z) = \widehat{g}(z)$ and $\widehat{g_{\alpha_1+\alpha_2}}(z) = \widehat{g_{\alpha_1} \star g_{\alpha_2}}(z)$. Thus, $g_1 = g$ and $g_{\alpha_1+\alpha_2} = g_{\alpha_1+\alpha_2}$. Moreover,

$$(\widehat{g}(z)) = a^{\alpha}\left(\frac{1 - e^{-z/2}}{z}\right)^{\alpha}(1 + a/be^{-z/2})^{\alpha} \to \left(\frac{a}{2}\right)^{\alpha}\left(1 + \frac{b}{a}\right)^{\alpha} = \left(\frac{a+b}{2}\right)^{\alpha}$$

and

$$(\widehat{g}(z))^{\alpha} = \widehat{g_{\alpha}}(z) = \int_0^{\infty} e^{-zu} g_{\alpha}(u)\, du \to \int_0^{\infty} g_{\alpha}(u)\, du$$

as $z \to 0$. It follows that

$$\int_0^{\infty} g_{\alpha}(u)\, du = \left(\frac{a+b}{2}\right)^{\alpha}.$$

Moreover, since $\widehat{g_{\alpha}}(z) = (\widehat{g}(z))^{\alpha} = \frac{a^{\alpha}}{z^{\alpha}}(1 - e^{-z/2})^{\alpha}$ is independent of $0 < \beta < \alpha$, it follows that the definition of g_{α} is independent of β.

Let

$$q_{\alpha}(u) = 2p_{\alpha}(2u) = \frac{2}{\Gamma(\alpha)}\sum_{j=0}^{[2u]} \psi_j^{-\alpha-1}(2u - j)^{\alpha-1}$$

be the convolution semigroup generated by $q(u) = 2\chi_{[0,\frac{1}{2}]}(u)$. Then

$$\widehat{q_{\alpha}}(z) = 2^{\alpha}(\frac{1 - e^{\frac{-z}{2}}}{z})^{\alpha}$$

and

$$\begin{aligned}\widehat{g_\alpha}(z) = \widehat{g}(z)^\alpha &= (\frac{a}{2})^\alpha 2^\alpha (\frac{1-e^{\frac{-z}{2}}}{z})^\alpha (1+\frac{b}{a}e^{\frac{-z}{2}})^\alpha \\ &= (\frac{a}{2})^\alpha 2^\alpha (\frac{1-e^{\frac{-z}{2}}}{z})^\alpha \sum_{k=0}^{\infty}(-1)^k \psi_k^{-\alpha-1}(\frac{b}{a})^k e^{\frac{-kz}{2}} \\ &= \left(\frac{a}{2}\right)^\alpha \sum_{k=0}^{\infty}(-1)^k \Psi_k^{-\alpha-1}\left(\frac{b}{a}\right)^k \widehat{q_\alpha}(z)e^{kz/2}.\end{aligned}$$

Since $\widehat{q_\alpha}(z)e^{\frac{-kz}{2}}$ is the Laplace transform of

$$q_\alpha(u-\frac{k}{2})\chi_{[\frac{k}{2},\infty)}(u) = \frac{2}{\Gamma(\alpha)}\sum_{j=0}^{[2u-k]}\Psi_j^{-\alpha-1}(2u-k-j)^{\alpha-1}\chi_{[\frac{k}{2},\infty)}(u).$$

If follows (at least formally) that

$$\begin{aligned}g_\alpha(u) &= \left(\frac{a}{2}\right)^\alpha \sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^\alpha (-1)^k \Psi_k^{-\alpha-1} q_\alpha\left(u-\frac{k}{2}\right)\chi_{[\frac{k}{2},\infty)}(u) \\ &= \left(\frac{a}{2}\right)^\alpha \sum_{k=0}^{[2u]}\left(\frac{b}{a}\right)^k (-1)^k \Psi_k^{-\alpha-1} q_\alpha\left(u-\frac{k}{2}\right) \\ &= \left(\frac{a}{2}\right)^\alpha \sum_{k=0}^{[2u]}\left(\frac{b}{a}\right)^k (-1)^k \Psi_k^{-\alpha-1}\frac{2}{\Gamma(\alpha)}\sum_{j=0}^{[2u-k]}\Psi_j^{-\alpha-1}(2u-k-j)^{\alpha-1} \\ &= \frac{a^\alpha}{2^{\alpha-1}}\sum_{k=0}^{[2u]}\left(\frac{b}{a}\right)^k (-1)^k \Psi_k^{-\alpha-1} p_\alpha(2u-k),\end{aligned}$$

Where p_α is the convolution semigroup generated by the function $p = \chi_{[0,1]}$. We summarize the observations above in the following proposition.

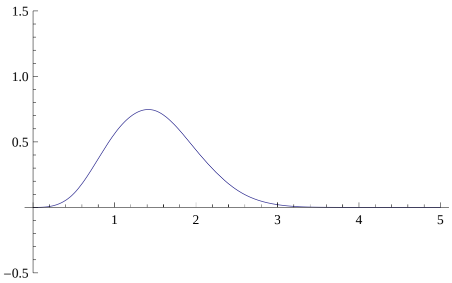
Proposition 1.4. *Let $a \geq |b| > 0$, $g(u) = a\chi_{[0,1/2]}(u) + b\chi_{[1/2,1]}(u)$, and $g_\alpha(u) = \frac{1}{\Gamma(\alpha)}\frac{a^\alpha}{2^{\alpha-1}}\sum_{k=0}^{[2u]}\left(\frac{b}{a}\right)^k(-1)^k\Psi_k^{-\alpha-1}\sum_{j=0}^{[2u-k]}\Psi_j^{-\alpha-1}(2u-k-j)^{\alpha-1}$. Then*

(i) $g_\alpha \star g_\beta = g_{\alpha+\beta}$

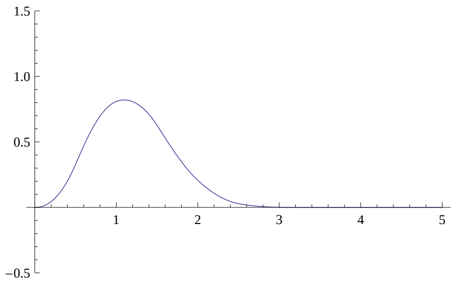
(ii) $g_1 = g$

(iii) $g_\alpha \in L^1(0,\infty)$, *and*

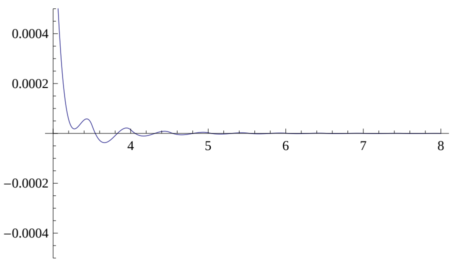
(iv) $\int_0^\infty g_\alpha(u)\,du = \left(\frac{a+b}{2}\right)^\alpha$.



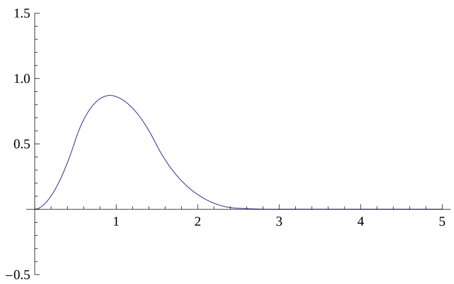
(a) $\alpha = 4.0$



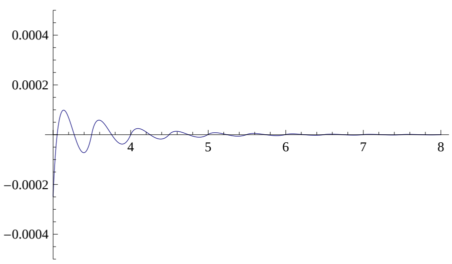
(b) $\alpha = 3.6$



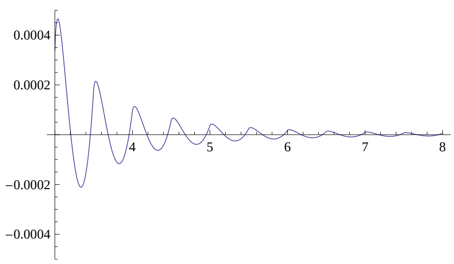
(c) $\alpha = 3.6$ (zoom)



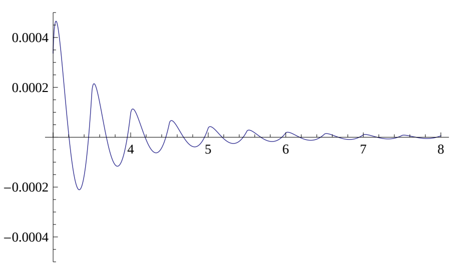
(d) $\alpha = 3.2$



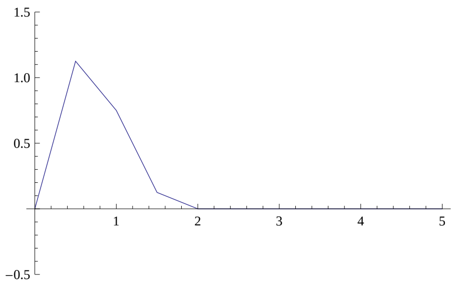
(e) $\alpha = 3.2$ (zoom)



(f) $\alpha = 2.8$

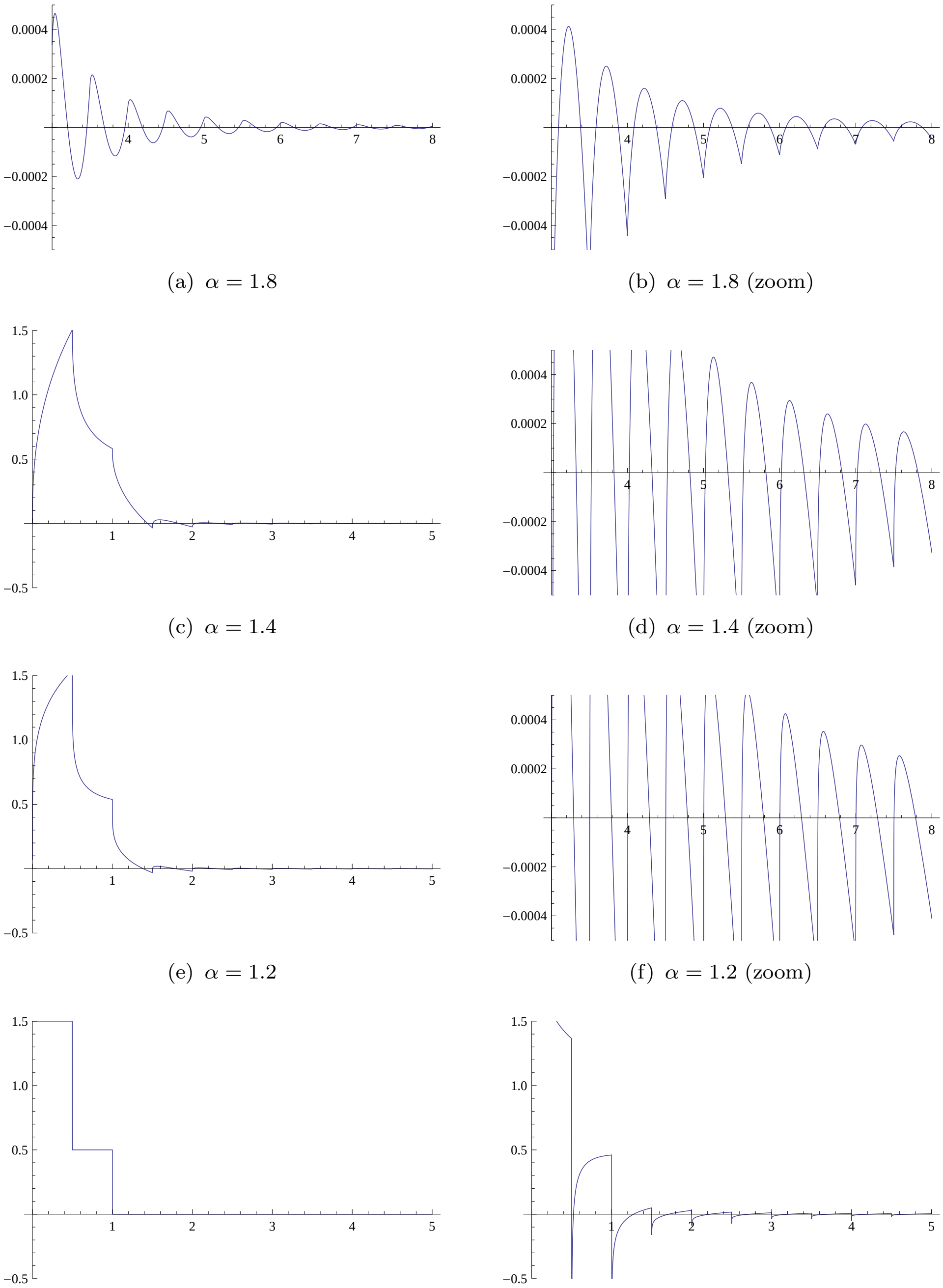


(g) $\alpha = 2.8$ (zoom)



(h) $\alpha = 2.0$

FIGURE 1.7. $(a\chi_{[0,1/2]} + b\chi_{(1/2,1]})^{\star\alpha}(u)$



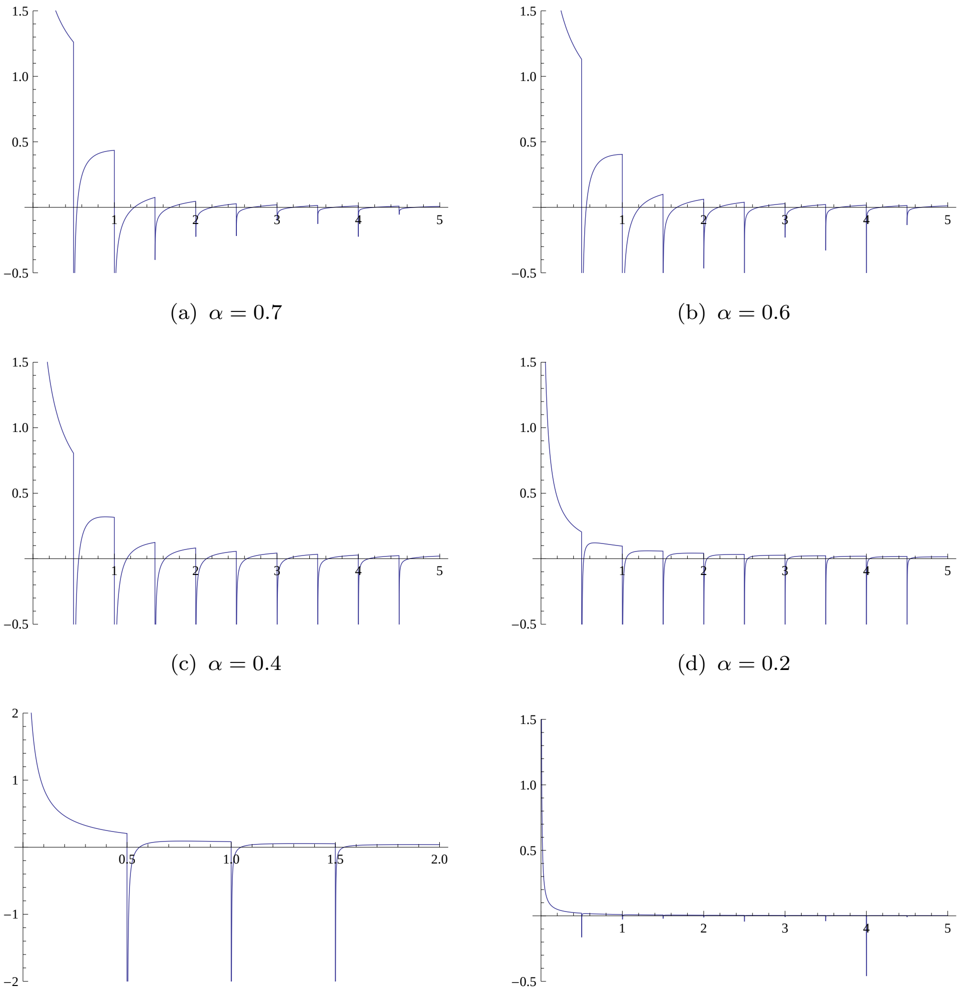
(a) $\alpha = 1.8$ (b) $\alpha = 1.8$ (zoom)

(c) $\alpha = 1.4$ (d) $\alpha = 1.4$ (zoom)

(e) $\alpha = 1.2$ (f) $\alpha = 1.2$ (zoom)

(g) $\alpha = 1.0$ (h) $\alpha = 0.8$

FIGURE 1.8. $(a\chi_{[0,1/2]} + b\chi_{(1/2,1]})^{\star\alpha}(u)$



(a) $\alpha = 0.7$ (b) $\alpha = 0.6$

(c) $\alpha = 0.4$ (d) $\alpha = 0.2$

(e) $\alpha = 0.1$ (f) $\alpha = 0.01$

FIGURE 1.9. $(a\chi_{[0,1/2]} + b\chi_{(1/2,1]})^{\star\alpha}(u)$

Like in the previous examples, the functions g_α are still "easily" computable, and we defer to Chapter 2 to explain a method by which we can graph the convolution semigroups $\{g_\alpha\}_{\alpha>0}$, efficiently if formulas such as those above are no longer available.

If the step function g is assumed to be positive (i.e., $a > 0, b > 0$), then the jump condition $|b| \leq a$ implies that the step function must be decreasing. This class of step functions will be considered next.

Theorem 1.5. *Let $g(u) = \sum_{j=0}^{N-1} a_j \chi_{[\frac{j}{N}, \frac{j+1}{N}]}(u)$ be a positive, decreasing step function (i.e, $a_0 \geq a_1 \geq \cdots \geq a_{N-1} > 0$). Then there exists a convolution semigroup $(g_\alpha)_{\alpha>0} \subset L^1(0,\infty)$ that is generated by g and that satisfies $\int_0^\infty g_\alpha(u)\, du = \left(\frac{a_0+a_1+\cdots+a_{N-1}}{N}\right)^\alpha$ for all $\alpha > 0$.*

Proof. For $z \in \mathbb{C}_+$,

$$\begin{aligned}
\widehat{g}(z) &= \frac{1}{z}\sum_{j=0}^{N-1} a_j[e^{-j/n} - e^{-(j+1)z/N}]\tilde{a_j}(e^{\frac{-z}{N}})^j \\
&= \frac{1}{z}a_{N-1}[1 - e^{-z/N}]\sum_{j=0}^{N-1} \tilde{a}_j(e^{\frac{-z}{N}})^j
\end{aligned}$$

where $\tilde{a}_j := \frac{a_j}{a_{N-1}}$ satisfy $\tilde{a}_0 \geq \tilde{a}_1 \geq \cdots \geq \tilde{g}_{N-1} = 1$. We may assume that for at least one j, $\tilde{a}_{j-1} > \tilde{a}_j$. Consider the polynomial

$$p(\omega) = \tilde{a}_0 + \tilde{a}_1\omega + \cdots + \tilde{a}_{N-2}\omega^{N-2} + \omega^{N-1}.$$

Then

$$\begin{aligned}
|(1-\omega)p(\omega)| &= |\tilde{a}_0 - (\tilde{a}_0 - \tilde{a}_1)\omega - \cdots - (\tilde{a}_{N-2} - 1)\omega^{N-1} - \omega^N| \\
&\geq \tilde{a}_0 - |(\tilde{a}_0 - \tilde{a}_1)\omega + (\tilde{a}_1 - \tilde{a}_2)\omega^2 + \cdots + (\tilde{a}_{N-2} - 1)\omega^{N-1} + \omega^N|.
\end{aligned}$$

Since for all ω with $|\omega| < 1$,

$$\begin{aligned}
|(\tilde{a}_0 - \tilde{a}_1)\omega + (\tilde{a}_1 - \tilde{a}_2)\omega^2 + \cdots + (\tilde{a}_{N-2} - 1)\omega^{N-1} + \omega^N| &< \\
|\tilde{a}_0 - \tilde{a}_1| + |\tilde{a}_1 - \tilde{a}_2| + \cdots + |\tilde{a}_{N-2} - 1| + 1 &= \tilde{a}_0
\end{aligned}$$

it follows that

$$|(1-\omega)p(\omega)| > 0$$

for all $|\omega| < 1$. Thus p has no roots in the open unit circle (see also [28, p. 107]) and we can write

$$\widehat{g}(z) = \frac{1}{z} a_{N-1}[1 - e^{\frac{-z}{N}}] \prod_{j=0}^{N-1} (d_j + e^{\frac{-z}{N}}),$$

where $-d_j$ are the roots of polynomial p with $|d_j| \geq 1$. Since $d_0 \cdot d_1 \cdots d_{N-1} = \tilde{a}_0$, it follows that

$$\begin{aligned} \widehat{g}(z) &= \frac{1}{z} a_{N-1} \tilde{a}_0 [1 - e^{\frac{-z}{N}}] \prod_{j=0}^{N-1} (1 + c_j e^{\frac{-z}{N}}) \\ &= \frac{1}{z} a_0 [1 - e^{\frac{-z}{N}}] \prod_{j=0}^{N-1} (1 + c_j e^{\frac{-z}{N}}), \end{aligned}$$

where $c_j := \frac{1}{d_j}$ and $|c_j| \leq 1$. Thus

$$\widehat{g}(z)^\alpha = \frac{1}{z^\alpha} a_0^\alpha [1 - e^{\frac{-z}{N}}]^\alpha \prod_{j=0}^{N-1} (1 + c_j e^{\frac{-z}{N}})^\alpha \tag{1.32}$$

$$= (\frac{\alpha_0}{N})^\alpha [\frac{N^{\alpha - \beta_0 - \beta_{N-1}} (1 - e^{\frac{-z}{N}})^\alpha}{z^{\alpha - \beta_0 - \cdots - \beta_{N-1}}}] \prod_{j=0}^{N-1} \frac{N^{\beta_j} (1 + c_j e^{\frac{-z}{N}})^\alpha}{z^{\beta_j}} \tag{1.33}$$

for some $0 < \beta_i$ with $\beta_0 + \beta_1 + \cdots + \beta_{N-1} < \alpha$.

Define $g_\alpha(u) = (\frac{a_0}{N})^\alpha q_{\alpha, \alpha - \beta_1, \cdots, -\beta_{N-1}, N} \star \prod_{j=0}^{N-1} \star h_{\alpha, \beta_j, N, c_j}$ where $q_{\alpha,\beta,N}$ and $h_{\alpha,\beta,\tau,c}$ are as in Lemma 1.3. Then, as above, $g_\alpha \in L^1(0,\infty)$, $\int_0^\infty g_\alpha(u)\, du = (\widehat{g}(z))^\alpha = \left(\frac{\alpha_0 + \cdots + \alpha_{N-1}}{N}\right)^\alpha$, $g_1 = g$, $g_{\alpha_1} \star g_{\alpha_2} = g_{\alpha_1 + \alpha_2}$ and the definition is independent of the particular choice of the numbers β_i. □

Theorem 1.5 suggests that every decreasing continuous function h with $supp[h] \subset [0,1]$ can be embedded in a convolution semigroup $(h_\alpha)_{\alpha > 0} \subset L^1(0,\infty)$. To this end, for a given $0 \leq h \in C[0,1]$, h decreasing, we select a sequence of decreasing step functions $g_n \geq 0$ such that $g_n \to h$ in the sup-norm, where

$$g_n(u) = \sum_{j=0}^{2^n - 1} a_{j_1 2^n} \chi_{[\frac{j}{2^n}, \frac{j+1}{2^n}]}(u).$$

By Theorem 1.5, g_n generates a convolution semigroup $(g_n^{\star\alpha})_{\alpha>0} \subset L^1(0,\infty)$ (see (1.33)). Suppose one could show that, for a fixed $\alpha > 0$, $g_n^{\star\alpha}$ is a Cauchy sequence in $L^1(0,\infty)$. Then there exists $h_\alpha \in L^1(0,\infty)$ such that $g_n^{\star\alpha} \to h_\alpha$ in $L^1(0,\infty)$ as $n \to \infty$. By the dominated convergence theorem

$$(\widehat{g_n}(z))^\alpha = \int_0^\infty e^{-zN} g_n^{\star\alpha}(u)\, du \to \int_0^\infty e^{-zu} h_\alpha(u)\, du.$$

Since $\widehat{g_n}(z) \to \widehat{h}(z)$ it follows that $(\widehat{h}(z))^\alpha = \int_0^\infty e^{-zu} h_\alpha(u)\, du$. In particular, $h_1 = h$ and $(h_\alpha)_{\alpha>0}$ is a convolution semigroup of L^1-functions.

Since it is not clear whether or not $g_n^{\star\alpha}$ is a Cauchy sequence in $L^1(0,\infty)$ for all $\alpha > 0$, we choose a different approach in the next section where we will show that each decreasing continuous function $h \geq 0$ with $supp[h] \subset [0,1]$ can be embedded into a convolution semigroup $(h_\alpha)_{\alpha>0}$ of distributions.

1.4 Convolution Semigroups Generated by Positive, Continuous, and Decreasing Functions with Compact Support.

In this section we take a different approach that concentrates more on the properties of $\widehat{h}$ and less on the properties of h. Since

$$\widehat{h}(z)^\alpha = e^{\alpha \ln\left(\widehat{h}(z)\right)},$$

the first question is whether or not $\ln\left(\widehat{h}(z)\right)$ is defined for all $\mathrm{Re}(z) \geq 0$. Only if $\widehat{h}(z)^\alpha$ can be defined as an analytic function in z for $\mathrm{Re}(z) \geq 0$ there is a chance that $(\widehat{h}(z))^\alpha$ is the Laplace transform of $h_\alpha \in L^1(0,\infty)$. It is easy to see that $z \to e^z$ maps the strip

$$\Omega := \{z = \{x, y\}, x \in \mathbb{R}, y \in (-\pi, \pi)\}$$

in a one-to-one fashion onto the sliced plane $\mathbb{C}\setminus(-\infty, 0]$. Thus, the inverse function

$$\log(z) := \log|z| + i\arg(z),$$

where $\arg(z) \in (-\pi, \pi)$, is well defined for $z \in \mathbb{C} \setminus (-\infty, 0]$.

Definition 1.6. *A subset D of the complex plane is log admissible if $D \cap (-\infty, 0] = \emptyset$. A function $f : \mathbb{C} \supset \Omega \to \mathbb{C}$ is called log admissible on Ω if $f(\Omega)$ is log admissible.*

Lemma 1.7. *Let $\Omega \subset \mathbb{C}$ be a bounded region and let f be analytic (and not constant) on Ω and continuous on $\bar{\Omega}$. Then $\partial f(\Omega) \subset f(\partial\Omega)$.*

Proof. Let $y \in \overline{f(\Omega)}$. Then there exists $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ such that $y = \lim_{n\to\infty} f(x_n)$. Since $\bar{\Omega}$ is compact, there exists a subsequence $(\bar{x}_n)_{n\in\mathbb{N}}$ and $x \in \bar{\Omega}$ such that $\tilde{x}_n \to x$ and $y = f(x)$. Thus $\overline{f(\Omega)} \subset f(\bar{\Omega})$ and, in particular, $\partial f(\Omega) \subset f(\bar{\Omega})$. Since $f(\Omega)$ is open, there can be no $c \in \Omega$ with $f(c) \in \partial f(\Omega)$. Thus $\partial f(\Omega) \subset f(\partial\Omega)$. $\square$

Example 1.8. Let $p(u) = \chi_{[0,1]}(u)$ and

$$\widehat{p}(z) = \frac{1 - e^{-z}}{z} = \int_0^\infty e^{-zu}\chi_{[0,1]}(u)du.$$

Then $p(0) = 1$,

$$|\widehat{p}(z)| \le \frac{2}{|z|}$$

for $(\mathrm{Re}(z) \ge 0)$, and $|\widehat{p}(z)| \le \int_0^\infty \chi_{[0,1]}(u)du = 1$ for $\mathrm{Re}(z) \ge 0$. Define $\mathbb{C}_+ := \{z : \mathrm{Re}(z) > 0\}$ and let $\epsilon > 0$. We show that

$$\widehat{p}(\mathbb{C}_+) \cap (-\infty, -\epsilon) = \emptyset.$$

To this end choose $n \in \mathbb{N}$ such that $\frac{2}{n} < \epsilon$ and consider

$$\Omega_n = \{z : \mathrm{Re}(z) > 0, |z| < n\}$$

and its complement Ω_n^c relative to $\mathbb{C}_+$; i.e., $\Omega_n^c = \{z : \mathrm{Re}(z) > 0 \text{ and } |z| \ge n\}$. If $z \in \Omega_n^c$ then $|\widehat{p}(z)| \le \frac{2}{|z|}$ and thus $\widehat{p}(\Omega_n^c) \cap (-\infty, -\epsilon) = \emptyset$. Since $|p(z)| \le 1$ for $z \in \mathbb{C}_+$ and since Ω_n is a bounded region, it follows that $\widehat{p}(\Omega_n)$ is again a bounded region whose boundary $\partial\widehat{p}(\Omega_n)$ satisfies

$$\partial\widehat{p}(\Omega_n) \subset \widehat{p}(\partial\Omega_n) = \widehat{p}(i[-n, n]) \cup \widehat{p}\{z : |z| = n,\ \mathrm{Re}(z) > 0\}.$$

Thus, if the closed curve $\widehat{p}(\partial\Omega_n)$ does not intersect $(-\infty,-\epsilon)$, then we know that the boundary $\partial p(\Omega_n)$ of the region $\widehat{p}(\Omega_n)$ does not intersect $(-\infty,-\epsilon)$ and, therefore, $\widehat{p}(\Omega_n)$ can not intersect $(-\infty,-\epsilon)$. Thus, if $\widehat{p}(i\mathbb{R}) \cap (-\infty,-\epsilon) = \emptyset$, then $\widehat{p}(\mathbb{C}_+) \cap (-\infty,-\epsilon) = \emptyset$ for all $\epsilon > 0$. Now,

$$\begin{aligned}\widehat{p}(ix) &= \frac{1-e^{-ix}}{ix} = \frac{ie^{-ix}-i}{x} = \frac{i\cos(x)+\sin(x)-i}{x} \\ &= i\frac{\cos(x)-1}{x} + \frac{\sin(x)}{x}.\end{aligned}$$

For $\widehat{p}(ix)$ to be in $(\infty,-\epsilon)$, it is necessary that $\mathrm{Im}(\widehat{p}(ix)) = \frac{\cos(x)-1}{x} = 0$, or $x = 2\pi n$ for some $n \in \mathbb{Z}$. However, we would then have $\mathrm{Re}(\widehat{p}(ix)) = \mathrm{Re}(\widehat{p}(i2\pi n)) = \frac{\sin(2n\pi)}{2n\pi} = 0$ for $n \neq 0$ or 1 for $n = 0$. This shows that $\widehat{p}(i\mathbb{R}) \cap (\infty,-\epsilon) = \emptyset$ for all $\epsilon > 0$. Hence

$$\widehat{p}(\mathbb{C}_+) \cap (-\infty,0) = \emptyset.$$

Suppose there exists $x+iyz \in \mathbb{C}_+$ with $\widehat{p}(z) = 0$. Then

$$1 = e^{-z} = e^{-x}(\cos(y)+i\sin(y)).$$

However, this is impossible if $x > 0$ and $y \in \mathbb{R}$. Thus

$$\widehat{p}(\mathbb{C}_+) \cap (-\infty,0] = \emptyset.$$

The method used in the previous example can be formulated into a useful lemma.

Lemma 1.8. *(a) Let $p \in L^1(0,\infty)$ be such that there exists $M > 0$ with $|z\widehat{p}(z)| \leq M$ for all $z \in \mathbb{C}_+$. If $\widehat{p}(i\mathbb{R}) \cap (-\infty,0) = \emptyset$, then $\widehat{p}$ is log admissible with respect to $\mathbb{C}_+$. (b) Let $p \in L^1(0,\infty)$ be piecewise continuous and bounded such that $p' \in L^1(0,\infty)$. Then there exists $M > 0$ such that $|z\widehat{p}(z)| \leq M$ for all $z \in \mathbb{N}_0$.*

Proof. For a proof of (a) see Example 1.8. For statement (b), observe that

$$\widehat{p}(z) = \int_0^\infty e^{-zu}p(u)du = \frac{1}{z}p(0) + \frac{1}{z}\int_0^\infty e^{-zu}p'(u)du$$

for $\mathrm{Re}(z) > 0$. It follows that $|z\widehat{p}(z)| \leq |p(0)| + \|p'\|_1 =: M$ for all $z \in \mathbb{C}_+$. $\square$

We saw in the previous section that positive, non-increasing functions h with $supp[h] \subset [0,1]$ might be embedded into a convolution semigroup $\{h_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$. To further investigate this topic, we now investigate the log admissibility of the Laplace transforms of such functions.

Lemma 1.9. *Let $h \in L^1(0,\infty)$ be positive, non-increasing, $L^1(0,\infty)$ with $supp[h] \subset [0,1]$, and let $y \in \mathbb{R}$. Then $\int_0^1 sin(ys)h(s)ds = 0$ if and only if $y = 0$.*

Proof. If $y = 0$, then $\int_0^1 sin(ys)h(s)ds = 0$. Now assume $y \neq 0$. Define $x_j := \frac{j\pi}{y}$ for $0 \leq j \leq j_0$, where j_0 is the smallest integer such that $\frac{j_0\pi}{y} \leq 1$. Furthermore, let $x_{j_0+1} = 1$. Then

$$\begin{aligned}
\int_0^1 h(s)sin(ys)ds &= \sum_{j=0}^{j_0-1}\int_{x_j}^{x_{j+1}} h(s)sin(ys)ds + \int_{x_{j_0}}^1 h(s)sin(ys)ds \\
&= \sum_{j=0}^{j_0-1}\int_{x_j}^{x_{j+1}+\frac{\pi}{y}} h(s)sin(ys)ds + \int_{x_{j_0}}^1 h(s)sin(ys)ds \\
&= \sum_{j=0}^{j_0-1}\int_0^{\pi/y} h(s+x_j)sin(ys+jy)ds + \int_{x_{j_0}}^1 h(s)sin(ys)ds \\
&= \int_0^{\pi/y} sin(ys)\sum_{j=0}^{j_0-1}(-1)^j h(s+x_j)ds + \int_{x_{j_0}}^1 h(s)sin(ys)ds
\end{aligned} \tag{1.34}$$

One notices that if $0 < y \leq \pi$, then the function $s \mapsto sin(ys)$ is strictly positive for $s \in (0,1)$. Therefore, because h is positive, $\int_0^1 h(s)sin(ys)ds > 0$. Now assume $y \geq \pi$, and therefore $j_0 \geq 1$. If j_0 is even, $j_0 - 1$ is odd. It then follows that

$$\sum_{j=0}^{j_0-1}(-1)^j h(s+x_j) = (h(s) - h(s+x_1)) + ... + (h(s+x_{j_0-2}) - (h(s+x_{j_{0-1}})) > 0.$$

Therefore,

$$\int_0^{\pi/y} sin(ys)\sum_{j=0}^{j_0-1}(-1)^j h(s+x_j)ds > 0.$$

Moreover, if j_0 is even, then $sin(ys) \geq 0$ for $s \in [x_{j_0}, 1]$. Thus, $\int_{x_0}^1 h(s)sin(ys)ds \geq 0$. Therefore,

$$\int_0^{\pi/y} sin(ys) \sum_{j=0}^{j_0-1} (-1)^j h(s+x_j)ds + \int_{x_{j_0}}^1 h(s)sin(ys)ds > 0.$$

Now let us assume that j_0 is odd. Then

$$\begin{aligned}
\int_0^1 h(s)sin(ys)ds &= \int_0^{\pi/y} sin(ys) \sum_{j=0}^{j_0-2} (-1)^j h(s+x_j)ds \\
&\quad + \int_0^{\pi/y} (-1)^{j_0-1} h(s+x_{j_0-1})sin(ys)ds \\
&\quad + \int_{x_{j_0}}^1 h(s)sin(ys)ds.
\end{aligned}$$

Moreover, because

$$\int_{x_{j_0}}^1 h(s)sin(ys)ds = -\int_0^{1-x_{j_0}} h(s+x_{j_0})sin(ys)ds,$$

we have that

$$\begin{aligned}
\int_0^1 h(s)sin(ys)ds &= \int_0^{\pi/y} sin(ys) \sum_{j=0}^{j_0-2} (-1)^j h(s+x_j)ds \\
&\quad + \int_0^{1-x_{j_0}} (h(s+x_{j_0-1}) - h(s+x_{j_0}))sin(ys)ds \\
&\quad + \int_{1-x_{j_0}}^1 h(s+x_{j_0}-1)sin(ys)ds \geq 0.
\end{aligned}$$

This completes the proof. □

Theorem 1.10. *Let $f \in L^1(0,\infty)$ with $supp[f] = [0,1]$. If f is strictly positive on $(0,1)$ and non-increasing on $(0,1)$, then $\widehat{f}$ is log-admissible with respect to $\mathbb{C}_+$.*

Proof. Let $z = x+iy \in \mathbb{C}_+$. Then $\mathrm{Re}(\widehat{f}(z)) = \int_0^1 e^{-xu}\cos(yu)f(u)\,du$ and $\mathrm{Im}(\widehat{f}(z)) = \int_0^1 e^{-xu}\sin(yu)f(u)\,du$. If $\mathrm{Im}(\widehat{f}(z)) = 0$, then it follows from Lemma 1.9, with $h(u) := e^{-xu}f(u)$ that $y = 0$. But then

$$\begin{aligned}\mathrm{Re}(\widehat{f}(z)) &= \int_0^1 e^{-xu}\cos(yu)f(u)\,du \\ &= \int_0^1 e^{-xu}f(u)\,du > 0.\end{aligned}$$

This shows that $\widehat{p}(\mathbb{C}_+) \cap (-\infty, 0] = \emptyset$. □

To proceed we observe that $|\widehat{f}(z)| \leq \|f\|_1$ for all $f \in L^1(0,\infty)$. Thus, if $\widehat{f}$ is log admissible with respect to $\mathbb{C}_+$, then $|\widehat{f(z)}^{\alpha}| \leq \|f\|_1^{\alpha}$ for all $z \in \mathbb{C}_+$ and the following theorem (see Theorem 2.5.1 in [1]) applies for $r_\alpha(z) := \frac{1}{z}\widehat{f}(z)^\alpha$ where $\alpha > 0$.

Theorem 1.11. *Let r be an analytic function on $\mathbb{C}_+$ such that $|r(z)| \leq \frac{M}{|z|}$ for some $M > 0$ and all $z \in \mathbb{C}_+$. Then, for all $b > 0$, there exists $p_b \in C(0,\infty)$ with $\sup_{u>0}\left|\frac{p_b(u)}{u^b}\right| < \infty$ such that $r(z) = z^b\widehat{p_b}(z)$ for all $z \in \mathbb{C}_+$.*

Corollary 1.12. *Let $f \in L^1(0,\infty)$ be such that $\widehat{f}$ is log admissible. Then for all $b, \alpha > 0$ there exists $p_{b,\alpha} \in C(0,\infty)$ with $\sup_{u>0}\left|\frac{p_{b,\alpha}(u)}{u^b}\right| < \infty$ such that*

$$(\widehat{f}(z))^\alpha = z^{b+1}\widehat{p_{b,\alpha}}(z) = \widehat{g_{b,\alpha}}(z),$$

where $g_{b,\alpha}$ is the $(b+1)^{th}$ distributional derivative of the continuous function $p_{b,\alpha}$. In particular, $g_{b,1} = f$ and $g_{b,\alpha_1} \star g_{b,\alpha_2} = g_{b,\alpha_1+\alpha_2}$.

Proof. The statement follows immediately from Lemma 1.11. For the proof of the fact that $z^{b+1}\widehat{p_{b,\alpha}}(z) = \widehat{g_{?\alpha}}(z)$ for the $(b+1)^{th}$ derivative $g_{b,\alpha}$ of $p_{b,\alpha}$ we refer the reader to [4] or [5]. □

Let $m_{b+1} : u \mapsto \frac{u^b}{\Gamma(b+1}$. Then by Proposition 1.10, and the remarks preceding it, it follows that $r_\alpha := m_{(b+1)\alpha}$ defines a convolution semigroup generated by m_b with $\widehat{r_\alpha}(z) = \frac{1}{z^{(b+1)\alpha}}$. Define

$$F\alpha(u) := (m_{(b+1)\alpha} \star g_{b,\alpha})(u) := \int_0^u \frac{(u-s)^{(b+1)\alpha}}{\Gamma((b+1)\alpha+1)} g_{b,\alpha}(u)\, du$$

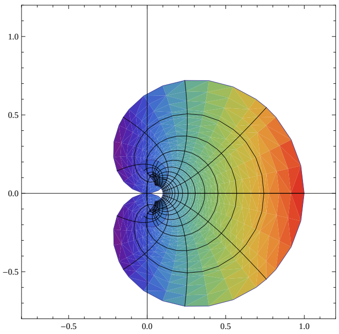
(the $(b+1)^{th}$ antiderivative of $g_{b,\alpha}$). Then

$$\begin{aligned}
\widehat{F_\alpha}(z) &= \widehat{m_{(b+1)\alpha}}(z)\widehat{g_{b,\alpha}}(z) \\
&= \frac{1}{z^{(b+1)\alpha}} \widehat{f}(z)^\alpha \\
&= \left(\frac{1}{z^{b+1}} \widehat{f}(z)\right)^\alpha
\end{aligned}$$

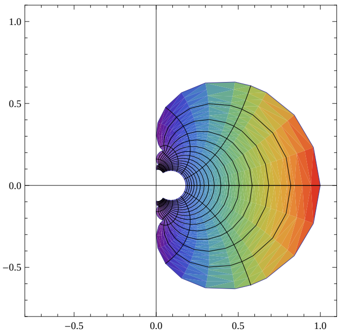
implies that F_α is the convolution semigroup generated by

$$F(u) = (m_{b+1} \star f)(u) = \int_0^u \frac{(u-s)^{(b+1)\alpha}}{\Gamma((b+1)\alpha+1)} g_{b,\alpha}(u)\, du$$

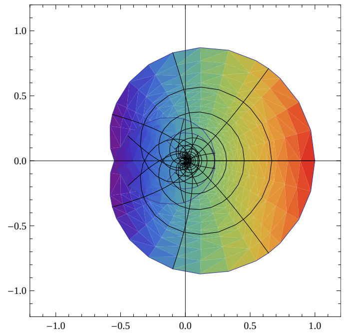
the $(b+1)^{th}$ anti-derivative of f. Notice that the functions $F_\alpha = m_{(b+1)(\alpha-1)} \star g_{b,\alpha}$ are continuous for all $\alpha > 1$. In Figure 1.10 we illustrate examples of functions which satisfy the hypotheses of Theorem 1.10 along with functions that do not admit log admissible Laplace transforms.



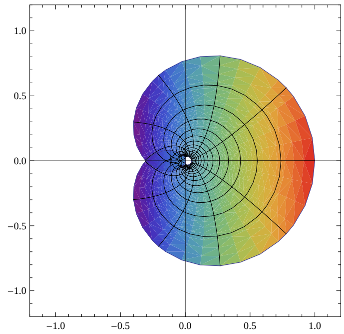
(a) $\widehat{\chi_{[0,1]}}(\mathbb{C}^+)$



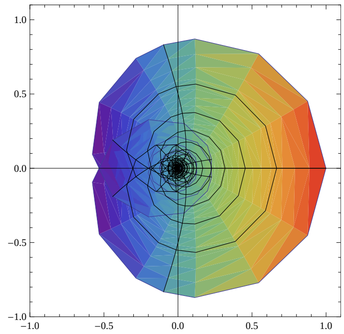
(b) $\widehat{s(1-s)\chi_{[0,1]}}(\mathbb{C}^+)$



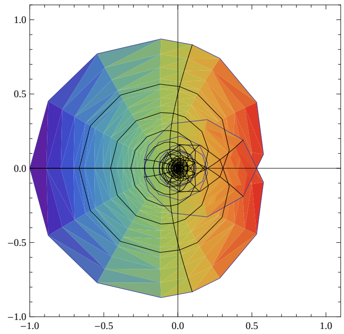
(c) $\widehat{4(1-8s)\chi_{[0,1]}}(\mathbb{C}^+)$



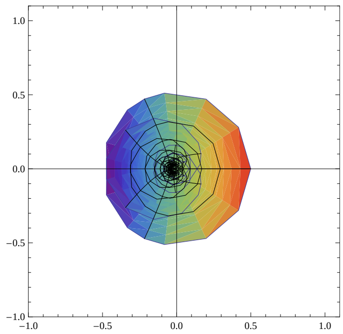
(d) $\widehat{6(s-s^2)\chi_{[0,1]}}(\mathbb{C}^+)$



(e) $\widehat{2s\chi_{[0,1]}}(\mathbb{C}^+)$



(f) $\widehat{-2s\chi_{[0,1]}}(\mathbb{C}^+)$



(g) $\widehat{(2s-1/2)\chi_{[0,1]}}(\mathbb{C}^+)$

FIGURE 1.10. The Image of the Laplace Transform

1.5 More Convolution Semigroups

In this section we will collect some basic mechanisms to generate "new" convolution semigroups out of given ones.

We start with the following observation. After the last section one might have the impression that only decreasing functions with compact support can generate convolution semigroups. This is not true. For example, let $(p_\alpha)_{\alpha>0}$ be the convolution semigroup generated by the characteristic function $p(0) = \chi_{[0,1]}(u)$ and let $q := p_{\alpha_0}$ $(\alpha > 0)$. Then q generates the convolution semigroup

$$\{q_\alpha\}_{\alpha>0} := \{p_{\alpha\alpha_0}\}_{\alpha>0}. \tag{1.35}$$

Clearly, as shown above, the generating functions $p_2, p_3, \ldots$ all have compact support, but are not decreasing.

Also if f generates a convolution semigroup $\{f_\alpha\}_{\alpha>0}$ and g generates a convolution semigroup $\{g_\alpha\}_{\alpha>0}$, then $f \star g$ generates a the semigroup

$$\{h_\alpha\}_{\alpha>0} := \{f_\alpha \star g_\alpha\}_{\alpha>0}. \tag{1.36}$$

In particular it follows from the following proposition that if f generates a convolution semigroup, then so does its anti-derivative $F := 1 \star f \ : u \mapsto \int_0^u f(s)ds$

Proposition 1.13. *Let $p(u) := \chi_{(0,\infty)}(u)$. Then p generates the convolution semigroup $\{p_\alpha\}_{\alpha>0}$ where $p_\alpha(u) := \frac{u^{\alpha-1}}{\Gamma(\alpha)}$ for $u > 0$.*

Proof. Since $\widehat{p}(z) = \frac{1}{z}$ and $\widehat{p_\alpha(0)} = \frac{1}{z^\alpha} = (\widehat{p}(z))^\alpha$, it follows that $p_1 = p$ and $p_{\alpha+\beta} = p_\alpha \star p_\beta$. ∎

Other convolution semigroups can be generated by the shifting and scaling of functions that generate convolution semigroups.

Proposition 1.14. *(Shifting and Scaling) Let $h \in L^1(0,\infty)$ be embedded in a convolution semigroup $(h_\alpha)_{\alpha>0} \subset L^1(0,\infty)$. Then the scaled function $g : u \rightarrow$*

$\tau h(\tau u)$ $(\tau > 0)$ can be embedded in the convolution semigroup $g_\alpha : u \to \tau h_\alpha(\tau u)$ and the shifted function

$$f : u \to h(u-a)\chi_{[a,\infty)}(u)$$

can be embedded in the convolution semigroup

$$f_\alpha : u \to h_\alpha(u-\alpha a)\chi_{[\alpha a,\infty)}(u)$$

Proof. Clearly $g_1 = g$ and

$$\widehat{g_\alpha \star g_\beta}(z) = \widehat{g_\alpha}(z)\cdot\widehat{g_\beta}(z) = \widehat{h_\alpha}(\frac{z}{\tau})\cdot\widehat{h_\beta}(\frac{z}{\tau}) = \widehat{h_{\alpha+\beta}}(\frac{z}{\tau}) = \widehat{g_{\alpha+\beta}}(z).$$

Thus $g_\alpha \star g_\beta = g_{\alpha+\beta}$. Similarly, $f_1 = f$ and

$$\begin{aligned}\widehat{f_\alpha \star f_\beta}(z) &= \widehat{f_\alpha}(z)\cdot\widehat{f_\beta}(z) &&= (e^{-\alpha a z}\widehat{h_\alpha}(z))(e^{-\beta a z}\widehat{h_\beta}(z))\\ &= e^{-(\alpha+\beta)az}\widehat{h_{\alpha+\beta}}(z) &&= \widehat{f_{\alpha+\beta}}(z).\end{aligned}$$

implies that $f_\alpha \star f_\beta = f_{\alpha+\beta}$.

□

Chapter 2

Numerical Approximation of Convolution Semigroups

In this chapter we will discuss numerical methods that allow us to approximate:

(i) Convolution semigroups generated by certain $p \in L^1(0, \infty)$ (as discussed in Chapter 1),

(ii) Operator semigroups induced by convolution semigroups.

To this end we will need the following prerequisites.

2.1 Prerequisites

Let $\mathcal{L}(X)$ denote Banach space of all bounded linear operators on a Banach space X. A semigroup T is a mapping from $[0, \infty)$ to the space $\mathcal{L}(X)$ which satisfies

$$\begin{cases} T(t+s) = T(t)T(s) \qquad t, s \in [0, \infty), \\ T(0) = I. \end{cases} \tag{2.1}$$

The mapping T is strongly continuous if the function $t \mapsto T(t)x$ defines a continuous function on $[0, +\infty)$ for each $x \in X$. We also say that the semigroup is of type (M, w) if there exist $M \geq 1$ and $w \in \mathbb{R}$ such that $||T(t)||_{\mathcal{L}(X)} \leq Me^{wt}$ for all $t \geq 0$. Finally, the generator of a strongly continuous semigroup is defined to be

$$Ax := \lim_{h \to 0^+} \frac{T(h)x - x}{h} \tag{2.2}$$

with domain $D(A)$ consisting of the elements of the Banach space X for which the limit (2.2) exists.

An important observation is to notice that if A generates a strongly continuous semigroup T on X and $x \in D(A)$ then $u(t) := T(t)x$ solves the abstract Cauchy problem

$$(ACP) \quad \begin{cases} u'(t) = Au(t) & (t \geq 0) \\ u(0) = x. \end{cases} \tag{2.3}$$

Example 2.1 (Shift Semigroup). Let Z be a complex Banach space and let $X := (C_0(\mathbb{R}, Z), \|\cdot\|_\infty)$ $(C_b([0,\infty), Z), \|\cdot\|_\infty)$ denote the Banach space of continuous functions from $\mathbb{R}^+$ to Z that vanish at infinity (that are bounded). For $t \geq 0$ and $x \in X$ consider the shift operator

$$T(t)x(\cdot) := x(\cdot + t).$$

Then $T(t) \in L(X)$ and $\|T(t)x\| \leq \|x\|$ for all $x \in X$. Moreover,

$$\|T(t)x - T(t_0)x\| \leq \sup_{0 \leq n} \|x(t+r) - x(t_0+r)\| + \sup_{r \geq n} \|x(t+r)\| + \sup_{r \geq n} \|x(t_0+r)\|.$$

Let $\epsilon > 0$. If $x \in C_0([0,\infty), Z)$, then there exists $N > 0$ such that the second and third terms are less than $\epsilon/3$ for each $t, t_0 \geq 0$. Since x is uniformly continuous on compact intervals, it follows that there exists $\delta > 0$ such that the first term is less than $\epsilon/3$ if $|t - t_0| < \delta$. Thus, if $x \in C_0([0,\infty), Z)$, then $t \mapsto T(t)x$ is continuous for all $x \in X$.

The situation is different if $X = C_b([0,\infty), Z)$. For example, let $z \in Z$ with $\|z\| = 1$ and let $x : r \mapsto e^{ir}z$. Then

$$\|T(t)x - T(t_0)x\| = \sup_{r \geq 0} |e^{i(t+r)^2} - e^{i(t_0+r)^2}| = 2$$

if $t \neq t_0$. Thus, the shift semigroup is not strongly continuous on $(C_b([0,\infty), Z))$.

The generator of the shift semigroup is, by definition,

$$Ax(\cdot) = \lim_{h \to 0^+} \frac{T(h)x(\cdot) - x(\cdot)}{h} = \lim_{h \to 0^+} \frac{x(\cdot + h) - x(\cdot)}{h} = x'(\cdot);$$

i.e., $A = D$, the differential operator on X with maximal domain

$$D(A) = \{x \in X : x \text{ is continuously differentiable and } x' \in X\}.$$

Let NBV_{loc} be the vector space of normalized functions from $(0, \infty)$ into $\mathbb{R}$ that are of bounded variation on each finite interval and let

$$NBV_0 := \{H \in NBV_{loc} : \|H\|_{BV} < \infty\}.$$

Then NBV_0 is a Banach algebra with Stieltjes-convolution

$$(H_1 \star_s H_2)(u) = \int_0^u H_1(u-s) dH_2(s)$$

as multiplication. Moreover, if $H(u) := \int_0^u h(s)ds$ for some $h \in L^1(0, \infty)$, then $\|H\|_{BV} = \|h\|_1$. Let

$$G_0 := \{f : f(z) = \int_0^\infty e^{zu} dH(u), H \in NBV_0\}.$$

If we define $\|f\|_G := \|H\|_{BV}$, then G_0 becomes a Banach algebra that is isometric isomorphic to NBV_0 (where $\Theta : NBV_0 \to G_0$ defined by $h \mapsto f_H$ with $f_H(z) := \int_0^\infty e^{zu} dH(u)$ is the isomorphism) (see [6],[13], and [18].)

If T is a bounded, strongly continuous semigroup on a Banach space X with generator A, and if $f \in G_0$ is given by $f(z) = \int_0^\infty e^{zu}\, dH(u)$ for some $H \in NBV_0$ and all $z \in \bar{\mathbb{C}}_-$, then one can define an operator $f(A)$ on X by

$$f(A) : x \mapsto \int_0^\infty T(t)x\, dH(t).$$

The following result justifies the use of the notation $f(A)$.

Theorem 2.1. *(Hille-Phillips Functional Calculus) Let A generate a strongly continuous semigroup T on a Banach space X of type $(M, 0)$. Then*

$$\psi : G_0 \to L(X), \quad \psi(f) := f(A)$$

is an algebra homomorphism and $\|f(A)\| \leq M\|H\|_{BV}$, where $H \in NBV_0$ is such that $f(z) = \int_0^\infty e^{zt}\, dH(t)$ for all $z \in \bar{\mathbb{C}}_-$.

As a first example demonstrating the usefulness of Theorem 2.1, let A generate a bounded, strongly continuous semigroup T, let $\lambda \in \mathbb{C}_+$, and

$$f(z) := \frac{1}{\lambda - z} \quad (z \in \mathbb{C}_-).$$

Then $f(z) = \int_0^\infty e^{zt}e^{-\lambda t}\,dt = \int_0^\infty e^{zt}h(t)\,dt = \int_0^\infty e^{zt}dH(t)$, where $h : t \to e^{-\lambda t} \in L^1(0,\infty)$, $H : t \to \int_0^t h(s)\,ds \in NBV_0$, and $\|H\|_{BV} = \|h\|_1 = \int_0^\infty |e^{-\lambda t}|\,dt = \frac{1}{Re\lambda}$. Thus, by Theorem 2.1,

$$R(\lambda, A)x := (\lambda I - A)^{-1}x = \int_0^\infty T(t)\,dH(t) = \int_0^\infty e^{-\lambda t}T(t)x\,dt \tag{2.4}$$

for all $x \in X$ and $\lambda \in \mathbb{C}_+$. In order to be able to state the inversion formulas for the Laplace transform we will use later to approximate our convolution semigroups, we have to discuss "rational approximations of semigroups" first. To this end, recall that

$$e^{tz} = \lim_{h\to\infty} r\left(\frac{tz}{h}\right)^n,$$

where $r(z) := \frac{1}{1-z}$. In order to find other rational functions r we observe that the Binomial Formula yields

$$\begin{aligned} |r(\tfrac{t}{n}z)^n - e^{tz}| &= |r(\tfrac{t}{n}z)^n - (e^{\frac{tz}{n}})^n| \\ &= |r(\tfrac{t}{n}z) - e^{\frac{t}{n}z}| \cdot |\textstyle\sum_{j=o}^{n-1}(e^{\frac{tz}{n}})^j \cdot r(\frac{tz}{n})^{n-1-j}|. \end{aligned}$$

Suppose r is $\mathcal{A}-$stable; i.e.,

$$|r(z)| \le 1 \quad \text{for all } z \in \bar{\mathbb{C}}_-. \tag{2.5}$$

For such r, since $|e^{\frac{t}{n}z}| \le 1$ for $z \in \bar{\mathbb{C}}_-$, it follows that the second term above is bounded by n and therefore

$$|r(\frac{t}{n}z)^n - e^{tz}| \le n|r(\frac{t}{n}z) - e^{\frac{t}{n}z}|.$$

Now, if

$$|r(z) - e^z| \le C \cdot |z|^{m+1} \tag{2.6}$$

for some $m > 0$ and for all $|z|$ sufficiently small, then

$$|r(\frac{t}{n}z)^n - e^{tz}| \le C \cdot t^{m+1} \cdot \frac{1}{n^m}|z|^{m+1} \tag{2.7}$$

and therefore $r(\frac{t}{n}z)^n \to e^{tz}$ for all $Rez \le 0$ and $t \ge 0$.

Definition 2.2. *A rational function r satisfying 2.5 and 2.6 is called a rational approximation of the exponential of order m.*

Example 2.4 (Rational Approximation of the Exponential)

(a) The Backward-Euler scheme

$$r(z) = \frac{1}{1-z}$$

is a rational approximation of the exponential of order $m = 1$. To see this observe that $|r(z)| = |\frac{1}{1-z}| \leq 1$ if $Rez \geq 0$ and that

$$\begin{aligned} \tfrac{1}{1-z} - e^z &= 1 + z + z^2 + \cdots - (1 + z + \tfrac{z^2}{2} + \cdots) \\ &= \tfrac{z^2}{2} + \tfrac{5z^3}{6} + \cdots \end{aligned}$$

or $|\frac{1}{1-z} - e^z| \leq C \cdot |z^2|$ for $|z| < 1$ small.

(b) The function

$$\begin{aligned} r(z) &= \tfrac{4z^3+60z^2+360z+840}{z^4-16z^3+120z^2-480z+840} \\ &= \textstyle\sum_{i=1}^4 \frac{B_i}{b_i - z} \end{aligned}$$

with $b_{1,2} \approx 3.212 \pm 4.773\, i$, $b_{3,4} \approx 4.787 \pm 1.567\, i$ and

$$B_i = \frac{4b_i^3 + 60b_i^2 + 360b_i + 84}{\prod_{k\neq i}(b_k - b_i)}$$

is a rational approximation of the exponential of order $m = 7$. For a proof, see [Reiser, Theorem III.2] and [Norsett, p. 481]. We mention this approximation here since it will be the basis for our methods later.

It is remarkable that the estimate (2.6) remains valid if z is replaced by the generator A of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of type $(M, 0)$. For a proof of the following result see [18] or [Reiser].

Theorem 2.3 (Brenner-Thomeé). *Let r be an $\mathcal{A}$-stable approximation of the exponential of order m and let A be the generator of a strongly continuous semigroup of type $(M, 0)$. Then*

$$\|r\left(\frac{t}{n}A\right)^n x - T(t)x\| \leq MCt^{m+1}\frac{1}{n^m}\|A^{m+1}x\|$$

for all $n \in \mathbb{N}$, $t \geq 0$, and $x \in D(A^{m+1})$, where the constant C is determined by r alone.

Recognizing the fundamental connection between the shift semigroup on spaces of continuous functions, Patricio Jara, Frank Neubrander, and Koray Özer used the Brenner-Thomeé theorem to establish a new inversion procedure for the Laplace transform. To describe their inversion procedure, recall from Example 2.1, that the shift semigroup

$$T(t)p(r) := p(t+r)$$

is a strongly continuous semigroup on $X := (C_0(\mathbb{R}+, Z), \|\cdot\|_\infty)$ with generator $A = \frac{d}{dr}$ and resolvent

$$R(\lambda, A)p = (\lambda I - A)^{-1}p = \int_0^\infty e^{-\lambda t}T(t)p\, dr$$

for all $\lambda \in \mathbb{C}_+$ and $p \in X$ Let $\lambda \in \mathbb{C}_+$, $z \in \mathbb{C}$, and $f(z) := \frac{1}{\lambda - z} = \int_0^\infty e^{zt}e^{-\lambda z}\, dt$. Then $\frac{d^n}{d\lambda^n}f(z) = \frac{(-1)^n n!}{(\lambda-z)^{n+1}} = \int_0^\infty e^{zt}(-t)^n e^{-\lambda t}\, dt$ and thus by the Hille-Phillips functional calculus

$$R(\lambda, A)^{n+1}p = \frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}R(\lambda, A)p = \frac{(-1)^n}{n!}\int_0^\infty e^{-\lambda t}(-t)^n T(t)p\, dt.$$

In particular, for the shift semigroup we obtain

$$T(t)p(0) = p(t) \qquad (t \geq 0) \tag{I}$$

$$R(\lambda, A)p(0) = \int_0^\infty e^{-\lambda t}p(t)\, dt = \widehat{p}(\lambda) \tag{II}$$

$$R(\lambda, A)^{n+1}p(0) = (-1)^n\int_0^\infty e^{-\lambda t}(-t)^n p(t)\, dt = \frac{(-1)^n}{n!}\widehat{p}^{(n)}(\lambda). \tag{III}$$

Moreover, since X is a space of continuous functions equipped with the sup-norm, we obtain from the Brenner Thomeé theorem that

$$\|r\left(\frac{t}{n}A\right)^n p(0) - T(t)p(0)\|_Z \leq \|r\left(\frac{t}{n}A\right)^n p - T(t)p\|_X \tag{2.8}$$

$$\leq MC\frac{t^{m+1}}{n^m}\|A^{m+1}x\| \tag{2.9}$$

if r is an $\mathcal{A}$-stable rational approximation of the exponential of order $m > 1$. Since r is $\mathcal{A}$-stable, all of the poles of r are in the right half plane and r has the partial fraction decomposition

$$r(z) = B_0 + \sum_{i=0}^{s}\sum_{j=1}^{r_i}\frac{B_{1,i,j}}{(b_i - z)^j}. \tag{2.10}$$

For example, the function

$$r(z) = \frac{4z^3 + 60z^2 + 360z + 840}{z^4 - 16z^3 + 120z^2 - 480z + 840}$$

is an $\mathcal{A}$-stable rational approximation of the exponential of order $m = 7$ and can be written as

$$r(z) = \frac{B_1}{b_1 - z} + \frac{B_2}{b_2 - z} + \frac{B_3}{b_3 - z} + \frac{B_4}{b_4 - z} \tag{2.11}$$

where the constants B_i, b_i can be computed symbolically and are approximated as in Example 2.4. Surely if r is as in (2.10), then $r(z)^n$ admits a partial fraction decomposition

$$r(z)^n = B_0 + \sum_{i=0}^{s}\sum_{j=1}^{r_i}\frac{B_{1,i,j}}{(b_i - z)^j}. \tag{2.12}$$

In particular if r is as in 2.11, then

$$r(z) = B_0 + \sum_{i=0}^{4}\sum_{j=1}^{n}\frac{B_{1,i,j}}{(b_i - z)^j}. \tag{2.13}$$

where the coefficients $B_{n,i,j}$ can be computed explicitly (for details see [13] or [16].) Thus by the Hille-Phillips functional calculus

$$
\begin{aligned}
r\left(\frac{t}{n}A\right)^n p &= B_0^n p + \sum_{i=0}^{s}\sum_{j=1}^{nr_i} B_{n,i,j}\left(b_i - \frac{t}{n}A\right)^{-j} p \qquad (2.14)\\
&= B_0^n p + \sum_{i=0}^{s}\sum_{j=1}^{nr_i} B_{n,i,j}\left(\frac{n}{t}\right)^j \left(\frac{b_i n}{t} - A\right)^{-j} p\\
&= B_0^n p + \sum_{i=0}^{s}\sum_{j=1}^{nr_i} B_{n,i,j}\left(\frac{n}{t}\right)^j R\left(\frac{b_i n}{t}, A\right)^{-j} p
\end{aligned}
$$

In particular, if r is as in 2.11, then

$$
r\left(\frac{t}{n}A\right)^n p = \sum_{i=0}^{4}\sum_{j=1}^{n} B_{n,i,j}\left(\frac{n}{t}\right)^j R\left(\frac{b_i n}{t}, A\right)^{-j} p. \qquad (2.15)
$$

Moreover, by (I)-(III) above

$$
r\left(\frac{t}{n}A\right)^n p = B_0^n p(0) + \sum_{i=0}^{s}\sum_{j=1}^{r_i n} B_{n,i,j}\left(\frac{n}{t}\right)^j \frac{(-1)^j}{(j-1)!}\widehat{p}^{(j-1)}\left(\frac{b_i n}{t}\right). \qquad (2.16)
$$

and if r is as in (2.11), then

$$
r\left(\frac{t}{n}A\right)^n p(0) = \sum_{i=0}^{4}\sum_{j=1}^{n} B_{n,i,j}\left(\frac{n}{t}\right)^j \frac{(-1)^{j-1}}{(j-1)!}\widehat{p}^{(j-1)}\left(\frac{b_i n}{t}\right). \qquad (2.17)
$$

The expressions in (2.16) and (2.17) are called rational inversions of the Laplace transform and are denoted by $\mathcal{L}_{r,n}^{-1}(\widehat{p})(t)$; i.e.,

$$
\mathcal{L}_{r,n}^{-1}(\widehat{p})(t) = B_0^n p(0) + \sum_{i=0}^{s}\sum_{j=1}^{r_i n} B_{n,i,j}\left(\frac{n}{t}\right)^j \frac{(-1)^j}{(j-1)!}\widehat{p}^{(j-1)}\left(\frac{b_i n}{t}\right). \qquad (2.18)
$$

Let

$$
E_r(n,t,p) := \|\mathcal{L}_{r,n}^{-1}(\widehat{p})(t) - p(t)\| \qquad (2.19)
$$

denote the error when replacing $p(t)$ by its approximation $\mathcal{L}_{r,n}^{-1}(\widehat{p})(t)$. Then by (2.8) and (2.16), if $p \in C_0(\mathbb{R}_+, Z)$, then

$$
E_r(n,t,p) \leq \frac{C}{n^m} t^{m+1} \|f^{(m+1)}\|_\infty \qquad (2.20)
$$

In the following, we will apply the rational approximation of the inverse Laplace transform with $m = 7$ and $n = 20$ to $\widehat{p}(z)^n$ in order to approximate $p^{\star n}(t)$ for $p(t) = \chi_{[0,1]}(t)$. Before we continue, we provide numerical evidence of the efficiency of the rational inversion formulas by applying them to $p(t) := \chi_{[0,1]}(t)$ and $g(t) := a\chi_{[0,1/2]}(t) + b\chi_{(1/2,1]}$ with $0 < b \leq |a|$. In both cases $p^{\star n}$ can be computed explicitly (see Chapter 1) and thus the error $E_r(n,t,p^{\star n})$ can be determined numerically.

2.2 Operator Semigroups Induced by Convolution Semigroups

In the 1950's, S. Bochner and W. Feller began the study of semigroups generated by fractional powers of the Laplace operator. Their goal was to study the diffusion equation

$$\begin{cases} u'(t) = -(-\Delta)^\alpha u(t) \qquad (t \geq 0), \\ u(0) = x, \end{cases} \tag{2.21}$$

for some $0 < \alpha < 1$ and the stochastic process associated to it, where Δ is the Laplacian operator. Their construction of the operator $-(-\Delta)^\alpha$, for $0 < \alpha < 1$, was based on the Hille-Phillips functional calculus and the fact that

$$\left(e^{-(-z)^\alpha}\right)^t = e^{-t(-z)^\alpha} = \int_0^\infty e^{zs} h_{\alpha,t}(s)\, ds,$$

where

$$h_{\alpha,t}(s) := \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\lambda} e^{-t\lambda^\alpha} d\lambda$$

if $s \geq 0$, and $h_{\alpha,t}(s) := 0$ for $s < 0$ is a probability density on $(0, \infty)$. Thus by the Hille-Phillips functional calculus $e^{-t(-\Delta)^\alpha} x$ defines an operator semigroup (where T_Δ is the semigroup generated by Δ.) R. S. Phillips in [25] showed that this construction is part of a more general framework, and extended the result, by using the Kolmogoroff-Levy representation theorem for infinitely divisible probability distributions. In this section, we extend Phillips' idea to arbitrary convolution semigroups.

Definition 2.4. *A function $p \in L^1(0, \infty)$ is said to generate a convolution semigroup $\{p_\alpha\}_{\alpha > 0} \subset L^1(0, \infty)$ if $p_1 = p$ and $p_{\alpha+\beta} = p_\alpha \star p_\beta$ for all $\alpha, \beta > 0$. The convolution semigroup is said to be of type K if $\|p_\alpha\|_1 \leq K^\alpha$ for all $\alpha > 0$ and some $K > 0$. Moreover, the semigroup $\{p_\alpha\}_{\alpha > 0} > 0$ is said to be continuous if $\|P_\alpha - H_0\|_1 \to 0$ as α tends to zero where $P_\alpha(u) := \int_0^u p_\alpha(s)\, ds$ and $H_0(u) := \chi_{(0,\infty)}(u)$.*

Proposition 2.5. *Let $\{p_\alpha\}_{\alpha>0} \subset L^1(0,\infty)$ be a continuous convolution semigroup of type K with $\int_0^\infty p_\alpha(u)\,du = 1$ for all $\alpha > 0$ and let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of type $(M,0)$ with generator A on some Banach space X. Then*

$$S(t) : x \mapsto \int_0^\infty T(s)x\, p_t(s)\, ds$$

defines a strongly continuous semigroup of type $(M, \ln(K))$. Moreover, if p_1 is a probability distribution describing the time t when the orbit $t \mapsto T(t)x$ is being observed for the first time, then

$$S(n)x = \int_0^\infty T(s)x\, p_n(s) ds$$

is the expected value of the orbit $t \mapsto T(t)x$ when it is being observed for the n^{th} time.

Proof. Let $\widehat{p_t}(z) = \int_0^\infty e^{zs} p_t(s)\,ds = \int_0^\infty e^{zs}\,dP_t(s)$ for $\mathrm{Re}(z) \leq 0$ where $P_t(s) := \int_0^s p_t(u)\,du$. Then $P_t \in NBV_0$ and $\|P_t\|_{BV} = \|p_t\|_1$. Thus $\widehat{p_t} \in G_0$ and the Hille-Phillips functional calculus can be applied. It follows that, for all $t > 0$,

$$\widehat{p_t}(A) : x \mapsto \int_0^\infty T(s)x\, dP_t(s) = \int_0^\infty T(s)x\, p_t(s)\, ds$$

is a bounded linear operator on X with

$$\|\widehat{P}_t(A)\| \leq M\|P_t\|_{BV} \leq MK^t = Me^{t\ln(K)}.$$

Moreover, since $\Lambda : G_0 \to L(X)$ is an algebra homomorphism, it follows from

$$\widehat{p_{t_1+t_2}}(z) = \widehat{p_{t_1} \star p_{t_2}}(z) = \widehat{p_{t_1}}(z)\widehat{p_{t_2}}(z)$$

that

$$\widehat{p_{t_1+t_2}}(A) = \widehat{p_{t_1}}(A) \circ \widehat{p_{t_2}}(A).$$

For $t > 0$, define $S(t) := \widehat{p_t}(A)$ and $S(0) := I$. Then

$$\begin{aligned} S(t)x - x &= \int_0^\infty T(s)x\, p_t(s)\, ds - x \\ &= \int_0^\infty T(s)x\, p_t(s)\, ds - \int_0^\infty T(s)x\, dH_0(s) \\ &= \int_0^\infty T(s)x\, d[P_t(s) - H_0(s)] \\ &= -\int_0^\infty p_t(s) - H_0(s)\, dT(s)x, \end{aligned}$$

since $P_t(\infty) - H_0(\infty) = 0$ and $P_t(0) - H_0(0) = 0$. Now let $x \in D(A)$. Then

$$S(t)x - x = -\int_0^\infty [p_t(s) - H_0(s)]T(s)Ax\, ds$$

and therefore $\|S(t)x - x\| \leq \|P_t - H_0\|_1 M \|Ax\|$ for all $t > 0$ and $x \in D(A)$. Since the operator family $\{S(t) - I\}_{0 \leq t \leq 1}$ is uniformly bounded and since $S(t)x \to x$ as $t \to 0$ for all x in the dense domain $D(A)$ (see [1] for a proof that $D(A)$ is dense in X), it follows that the semigroup $\{S(t)\}_{t \geq 0}$ is strongly continuous in $t = 0$. Again by elementary semigroup theory, it follows that $\{S(t)\}_{t \geq 0}$ is strongly continuous in $[0, \infty)$. For a proof of the other statements, see the introduction. □

Although all of the numerical evidence supports the claim that the convolution semigroups generated by positive, decreasing step functions satisfy

$$\|p_\alpha - H_0\|_1 \to 0$$

as $\alpha \to 0$, and are therefore continuous convolution semigroups, we were unable to show this. However, for the purpose of approximating the expected values

$$S(n)x = \int_0^\infty T(s)xp^{\star n}(s)\, ds$$

this is irrelevant. The only thing needed is that the operators $S(n)$ are well-defined - which they are by the Hille-Phillips functional calculus.

2.3 The Newton-Cotes Quadrature Methods in Banach Spaces

In this section, we present a method by which we may extend the typical Newton-Cotes Quadrature methods to Banach space valued functions. We will show that the error estimates normally obtained via Newton-Cotes also extend the way of Banach space valued functions. The Newton Cotes formulas represent a method for approximating integrals of the form $\int_a^b f(r)\,dr$ by summations of the form $\sum_{i=0}^{n} a_i f(r_i)$, where the coefficients a_i are given by

$$a_i := \int_a^b L_i(r)dr \text{ with } L_i(r) := \prod_{j\neq 1, j=0} \frac{(r-r_j)}{r_i - r_j},$$

and $r_i := a + \frac{b-a}{n} i$ for $i \in \{0, ..., n\}$. In order to demonstrate the method by which we may extend Newton-Cotes, we now present the following theorem from classical error analysis. For a proof of this theorem, for numerical functions, see [9].

Theorem 2.6. *Let $N \in \mathbb{N}$, and suppose that $f : [a,b] \to \mathbb{C}$ is $(N+2)-$times continuously differentiable if N is even. If N is odd, however, we assume that $f : [a,b] \to \mathbb{C}$ is $(N+1)-$times continuously differentiable. Furthermore, let us suppose that $\sum_{i=0}^{N} a_i f(r_i)$ is the $(N+1)-$point closed Newton Cotes formula. Then there exists $\xi \in (a,b)$ such that*

$$\int_a^b f(r)dr = \sum_{i=0}^{N} a_i f(r_i) + \frac{(b-a)^{N+3}}{(N+2)!} f^{(N+2)}(\xi) \int_0^N t^2(t-1)\cdots(t-N)dt \quad (2.22)$$

if N is even. If N is odd, then it follows similarly that

$$\int_a^b f(r)dr = \sum_{i=0}^{N} a_i f(r_i) + \frac{(b-a)^{N+2}}{(N+2)!} f^{(N+1)}(\xi) \int_0^N t^2(t-1)\cdots(t-N)dt. \quad (2.23)$$

We note here that because there is no Mean Value Theorem for Banach Space valued functions, the error estimates included in (2.22) and (2.23) will, in general, no longer be valid for functions with range in a Banach space X. However, the following estimates remain valid (see [22])

Corollary 2.7. *Let X be a Banach space, and let $f : [a, b] \to X$ be $(N+2)-$times continuously differentiable. If $\sum_{i=0}^{N} a_i f(r_i)$ is the $(N+1)-$point closed Newton Cotes Formula, then*

$$\left\| \int_a^b f(r)dr - \sum_{i=0}^{N} a_i f(r_i) \right\| \leq \frac{(b-a)^{N+3}}{(N+2)!} \|f^{(N+2)}\|_{L^\infty(a,b)} \left| \int_0^N t^2(t-1)\cdots(t-N)dt \right| \tag{2.24}$$

if N is even. On the other hand, if N is odd, then

$$\left\| \int_a^b f(r)dr - \sum_{i=0}^{N} a_i f(r_i) \right\| \leq \frac{(b-a)^{N+2}}{(N+1)!} \|f^{(N+1)}\|_\infty \left| \int_0^N t(t-1)\cdots(t-N)dt \right| . \tag{2.25}$$

Proof. We will first consider the case where N is even. To this end let X^* denote the dual space of X, and let $\mu \in X^*$ with $\|\mu\| = 1$. Then there exists ξ_μ such that

$$\begin{aligned} \left| \left\langle \int_a^b f(r)\, dr - \sum_{i=0}^{N} a_i f(r_i), \mu \right\rangle \right| &= \left| \int_a^b \langle f(r), \mu\rangle \, du - \sum_{i=0}^{N} a_i \langle f(r_i), \mu\rangle \right| \\ &\leq \left| \frac{(b-a)^{N+3}}{(N+2)!} \langle f^{(N+2)}(\xi_\mu, \mu\rangle \right| \left| \int_0^N t^2(t-1)\cdots(t-N)dt \right| \\ &\leq \frac{(b-a)^{N+3}}{(N+2)!} \|f^{(N+2)}\|_{L^\infty} \left| \int_0^N t^2(t-1)\cdots(t-N)dt \right| . \end{aligned}$$

If we now note that $\sup_{\mu \in X^*, \|\mu\|=1} |\langle x, \mu\rangle| = \|x\|$, then the preceding inequalities immediately justify (2.24). The proof of (2.25) is nearly identical. $\square$

In the case where $n = 1$, Theorem 2.6 becomes the well known trapezoidal rule. In other words, if $N = 1$, then $\int_a^b f(r)\, dr = \frac{b-a}{2}(f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\xi)$ for some ξ with $a < \xi < b$, if f is scalar valued, and

$$\left\| \int_a^b f(r)\, dr - \frac{b-a}{2}(f(a) + f(b)) \right\| \leq \frac{(b-a)^3}{12} \|f''\|_{L^\infty} \tag{2.26}$$

if f takes its values in a Banach space.

2.4 Approximation of Operators Induced by the Hille-Phillips Functional Calculus

Let A generate a strongly continuous semigroup T of type $(M,0)$ on a Banach space X. and let $p \in L^1(0,\infty)$. Then $P : u \mapsto \int_0^u p(s)\,ds \in NBV_0$ with $\|P\|_{BV} = \|p\|_1$. Thus the function f defined by

$$f(z) := \int_0^\infty e^{zu} p(u)\,du = \int_0^\infty e^{zu}\,dP(u)$$

for $z \in \bar{\mathbb{C}_-}$ is in G_0, and thus, by the Hille-Phillips functional calculus, the operator

$$f(A) : x \mapsto \int_0^\infty T(u)x\,p(u)\,du$$

is in $L(X)$ and $\|f(A)\| \leq \|p\|_1$. Since $p \in L^1(0,\infty)$, there exists, for all $\epsilon > 0$, an $a > 1$ such that $\int_0^\infty |p(u)|\,du < \frac{\epsilon}{M}$. Thus

$$\|f(A)x - \int_0^a T(u)x\,p(u)\,du\| \leq \|\int_a^\infty T(u)x\,p(u)\,du\| \leq \epsilon\|x\|.$$

If $\sum_{i=0}^N a_i f(r_i)$ is the $(N+1)-$point closed Newton Cotes formula on $[0,a]$ for $g(u) := T(u)x\,p(u)$ sufficiently smooth (i.e., if $x \in D(A^{N+2})$ and $p \in C^{N+2}(0,\infty)$), then

$$\begin{aligned}
\|f(A)x - \sum_{i=0}^N a_i T(r_i)x\,p(r_i)\| &\leq \|f(A)x - \int_0^u T(u)x\,p(u)\,du\| \\
&\quad + \|\int_0^a T(u)xp(u)\,du - \sum_{i=0}^N a_i T(r_i)x\,p(r_i)\| \\
&\leq \begin{cases} \epsilon\|x\| + C_n \frac{a^{N+3}}{(N+2)!}\|g^{(N+2)}\|_\infty & \text{if } N \text{ is even,} \\ \epsilon\|x\| + \tilde{C}_n \frac{a^{N+2}}{(N+1)!}\|g^{(N+1)}\|_\infty & \text{if } N \text{ is odd.} \end{cases} \\
&=: M_{\epsilon,N,x}
\end{aligned}$$

where $C_N := \int_0^N t^2(t-1)\cdots(t-N)\,dt$, $\tilde{C_N} \int_0^N t(t-1)\cdots(t-N)\,dt$, and $g(u) = p(u)T(u)x$. We remark here that

$$
\begin{aligned}
|g^{(N)}(u)| &= |\sum_{j=0}^{N} \binom{N}{j} T(u) x^{(N-j)} p^{(j)}(u)| && (2.27)\\
&= |\sum_{j=0}^{N} \binom{N}{j} T(u) A^{N-j} x \, p^{(j)}(u)| && (2.28)\\
&= sum_{j=0}^{N} \binom{N}{j} T(u) \|A^{N-j} x\| \, \|p^{(j)}(u)\|_{\infty}. && (2.29)
\end{aligned}
$$

Now consider the rational inversion $\mathcal{L}_{r,n}^{-1}(\widehat{p})(t)$ of the Laplace transform of order $m > 1$ defined in (2.18); i.e.,

$$
\mathcal{L}_{r,n}^{-1}(\widehat{p})(t) = B_0^n p(0) + \sum_{i=0}^{s} \sum_{j=1}^{r_i n} B_{n,i,j} \left(\frac{n}{t}\right)^j \frac{(-1)^j}{(j-1)!} \widehat{p}^{(j-1)} \left(\frac{b_i n}{t}\right). \qquad (2.30)
$$

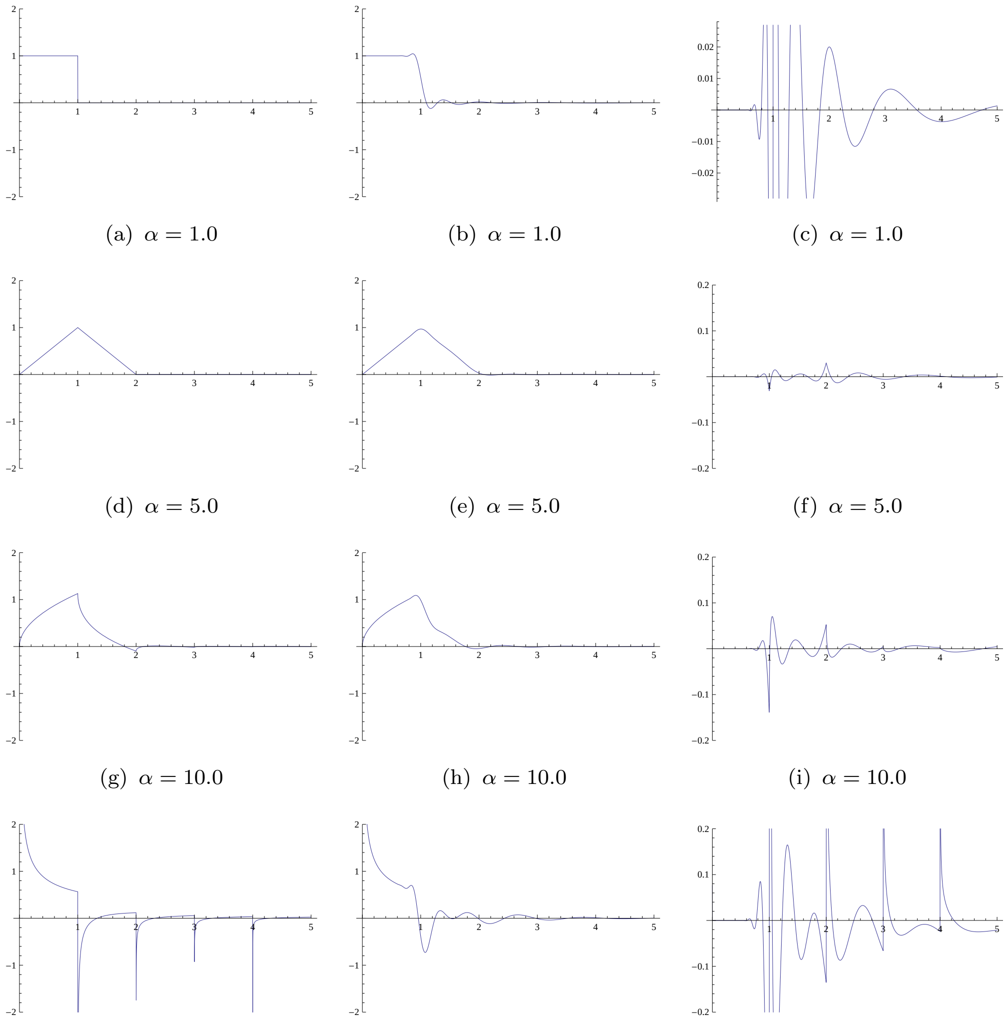
Then

$$
\begin{aligned}
\|f(A)x - \sum_{i=0}^{N} a_i T(r_i) x \mathcal{L}_{r,n}^{-1}(\widehat{p})(r_i)\| &\leq \|f(A)x - \sum_{i=0}^{N} a_i T(r_i) x \, p(r_i)\| \\
&\quad + \|\sum_{i=0}^{N} a_i T(r_i) x \left[p(r_i) - \mathcal{L}_{r,n}^{-1}(\widehat{p})(r_i)\right]\| \\
&\leq M_{\epsilon,N,x} + M\|x\| \sum_{i=0}^{N} a_i E_r(n, r_i, p) \\
&\leq M_{\epsilon,N,x} + \frac{CM}{n^m} \|x\| \|p^{(m+1)}\|_{\infty} \sum_{i=0}^{N} a_i r_i^{m+1}
\end{aligned}
$$

where $M_{\epsilon,N,x}$ and $E_r(n, r_i, p)$ are defined as above. Finally by the Brenner-Thomeé theorem it follows that

$$
\begin{aligned}
&\|f(A)x - \sum_{i=0}^{N} a_i r(\frac{r_i}{n} A)^n x \, \mathcal{L}_{r,n}^{-1}(\widehat{p})(r_i)\| \\
&\qquad \leq \|f(A)x - \sum_{i=0}^{N} a_i r(\frac{r_i}{n} A)^n x \, \mathcal{L}_{r,n}^{-1}(\widehat{p})(r_i)\|
\end{aligned}
$$

We remark that the estimate becomes significantly better if we split the interval $[0, a]$ into finitely many pieces of maximal length $0 < n \leq h$ and apply the Newton Cotes formulas on each piece separately. Then the estimates we obtained



(a) $\alpha = 1.0$ (b) $\alpha = 1.0$ (c) $\alpha = 1.0$

(d) $\alpha = 5.0$ (e) $\alpha = 5.0$ (f) $\alpha = 5.0$

(g) $\alpha = 10.0$ (h) $\alpha = 10.0$ (i) $\alpha = 10.0$

(j) $\alpha = 100.0$ (k) $\alpha = 100.0$ (l) $\alpha = 100.0$

FIGURE 2.1. $\chi_{[0,1]}(u)^{\star\alpha}(u)$: explicit (left), approximation (middle), error (right).

above remains essentially the same with the constant a being replaced by h. In the following we will test our results by comparing the "true" value of

$$s(u)x = \int_0^\infty T(u)x\, p^{\star n}(u)\, du$$

with $p(u) = \chi_{[0,1]}(u)$ and $T(u)x := x(u+1)$ with its approximation, where we take r to be the $\mathcal{A}$-stable rational approximation of the exponential of order $m = 7$ (see Example 2.4) with $n = 20$. If $p(u) = \chi_{[0,1]}(u)$, then as was shown in Chapter 1 $p^{\star n}$ has support in $[0, n]$, and therefore $S(n)x(r) = \sum_{j=0}^{n-1}(-1)^j \int_0^n x(u+r)\, du$ can be easily computed for "basic" functions x.

2.5 The Calculation of the Expected Values of the Shift Semigroup upon the n^{th} Observation

Throughout this section we will consider the semigroup, $\{T(t)\}_{t\geq 0}$ and the Banach space X defined in Example 2.1. That is, we will assume from here on that the semigroup $\{T(t)\}_{t\geq 0} : C_{ub}[0,\infty) \mapsto C_{ub}[0,\infty)$ is defined by $T(t)x(\cdot) \mapsto x(\cdot + t)$, where $C_{ub}[0,\infty)$ is the Banach space of all bounded, uniformly continuous from $[0,\infty)$ into $\mathbb{C}$. In what follows we will concern ourselves with the calculation of the expected value of the system $T(t)$ upon being observed for the n^{th} time where the probability density describing the time upon which the n^{th} observation is made is given by some function f_n. We begin our discussion with the density defined by $p_n(s) := \chi_{[0,1]}(s)$. We will then explore a slightly less trivial case given by a two-step function. This section will be concluded with an example demonstrating the methods outlined in the previous section; i.e., we will investigate a case where the function f_n is approximated using the rational inversion methods for the Laplace transform introduced in Section 2.4.

2.5.1 Westphal's Example Revisited

Let $\{T(t)\}_{t\geq 0}$ be the shift semigroup; i.e., let $\{T(t)\}_{t\geq 0} : C_{ub}[0,\infty) \mapsto C_{ub}[0,\infty)$ be defined by $T(t)x(\cdot) \mapsto x(\cdot + t)$. Moreover let us suppose that the probability density describing the time upon which the n^{th} observation is made is given by $p_n(s)$, which is defined as in (1.7). That is

$$p_n(s) := \frac{1}{\Gamma(n)} \sum_{j=0}^{[u]} (-1)^j \binom{n}{j} (u-j)^{n-1}.$$

Then the expected value of the system $T(s)$ upon being observed for the n^{th} time, which we denote by $E(n)x$, is known to be

$$E(n)x = \int_0^\infty T(s)x\, p_n(s)\, ds = \int_0^\infty T(s)x\, p^{\star n}(s)\, ds. \tag{2.31}$$

As we have shown, we may explicitly calculate the convolution powers $p^{\star n}$ even for the case where n is non-integral. We note here that although the non-integral case may be "mathematically interesting", we concern ourselves with only the cases which have a simple, physical interpretation. Now let us assume that $T(s)x$ is the state of an evolutionary system at time $t \geq 0$ where $x(r) := re^{-r}$ for $0 \leq r < \infty$. Using the formula obtained in (1.7), we may now illustrate the expected states of the system in the following figures.

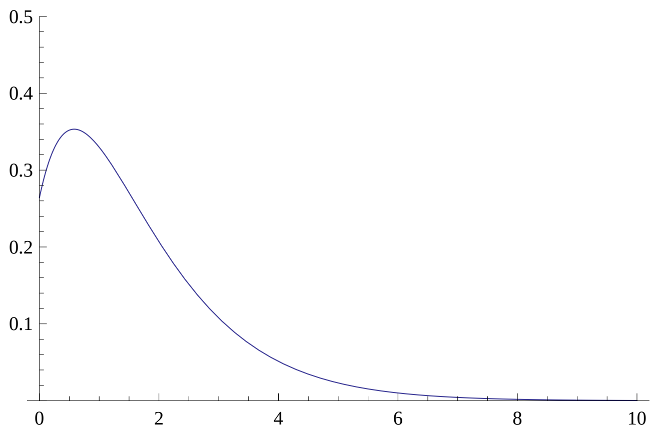


FIGURE 2.2. Expected Value upon First Observation

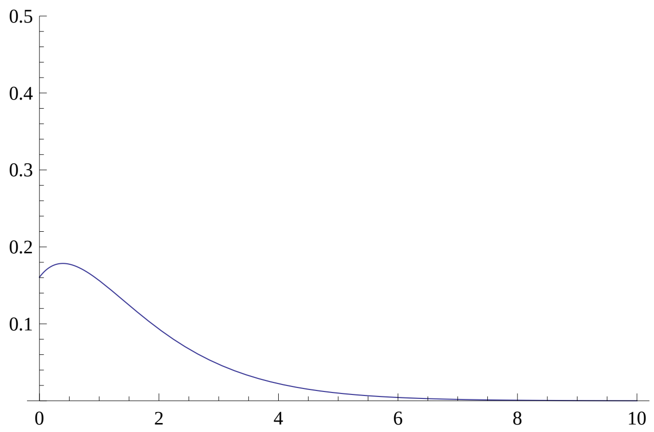


FIGURE 2.3. Expected Value upon Second Observation

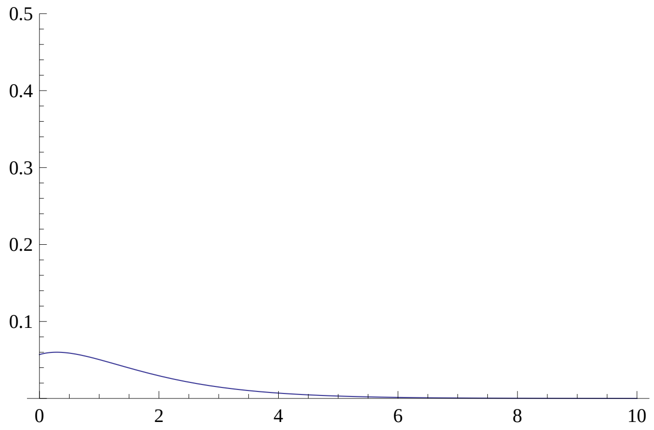


FIGURE 2.4. Expected Value upon the Third Observation

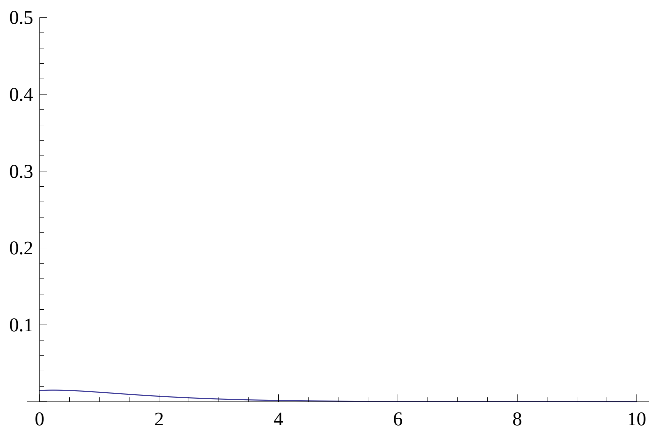


FIGURE 2.5. Expected Value upon the Fourth Observation

We will now consider the same system, however, we will introduce a different method for computing the expected value. That is, we will now employ the rational approximation methods for the inversion of the Laplace transform discussed earlier. Our goal here is to illustrate the accuracy of the methods, and to provide a transition into the next section where we illustrate the expected values of systems which are at times given by densities of a far less trivial nature. That is, we no longer use explicitly defined functions to represent convolution powers. We urge the reader to compare the following images with those found in Figure 2.2, Figure 2.3, Figure 2.4, and Figure 2.5.

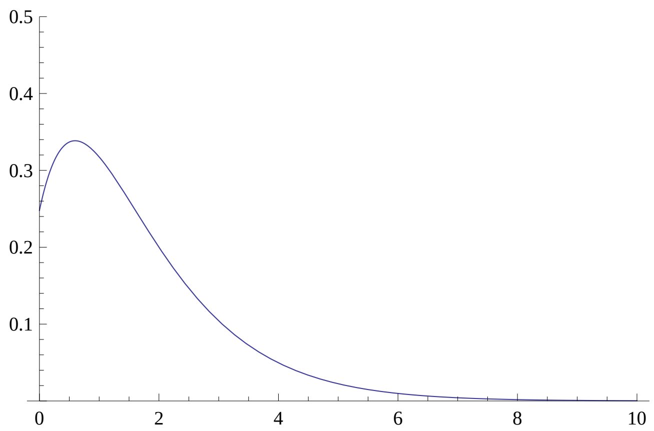


FIGURE 2.6. Expected Value upon First Observation (Approximation)

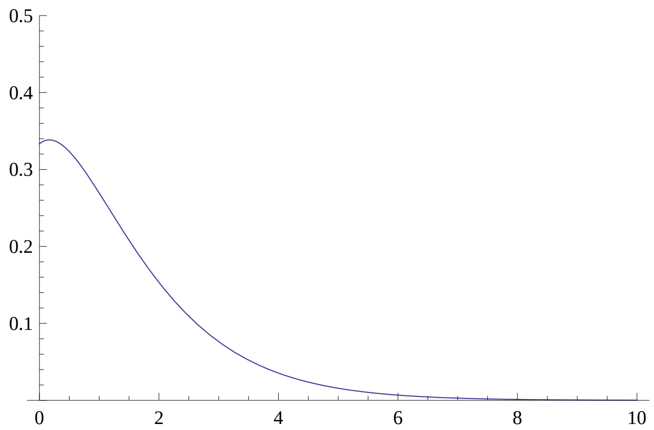


FIGURE 2.7. Expected Value upon Second Observation (Approximation)

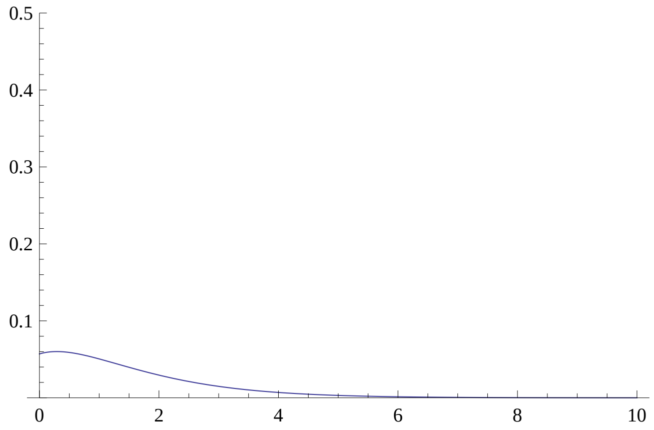


FIGURE 2.8. Expected Value upon the Third Observation (Approximation)

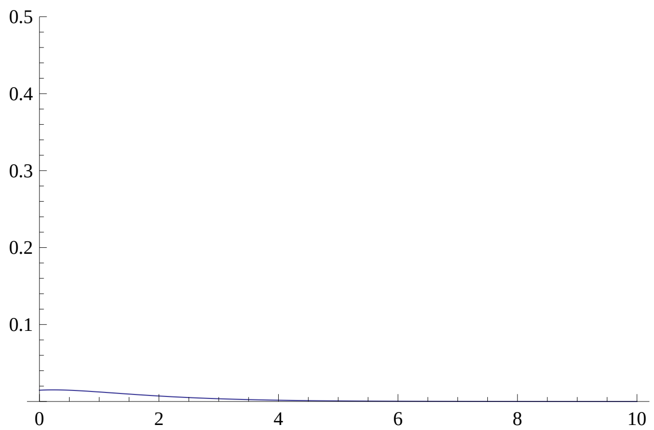


FIGURE 2.9. Expected Value upon the Fourth Observation (Approximation)

We now provide illustrations depicting the differences between the graphs obtained explicitly and those based on the rational approximations described previously. Note the considerable change in scale in Figures 2.12 and 2.13.

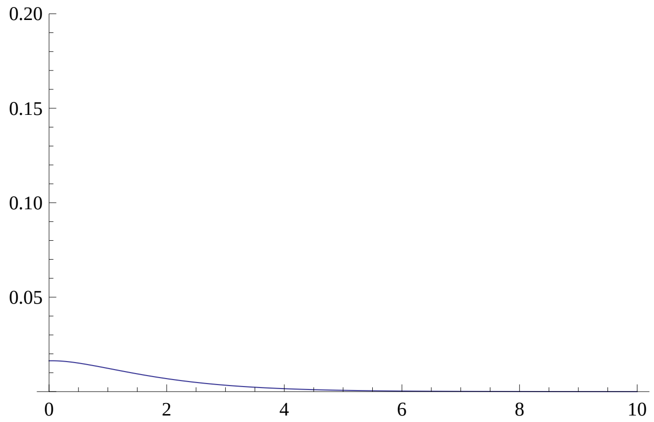


FIGURE 2.10. Error in Expected Value upon First Observation

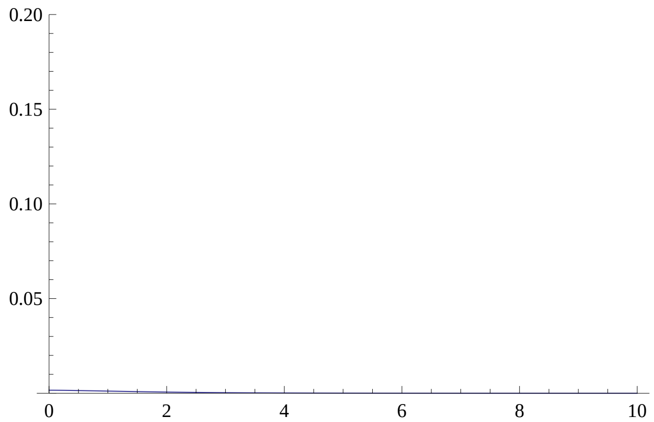


FIGURE 2.11. Error in Expected Value upon Second Observation

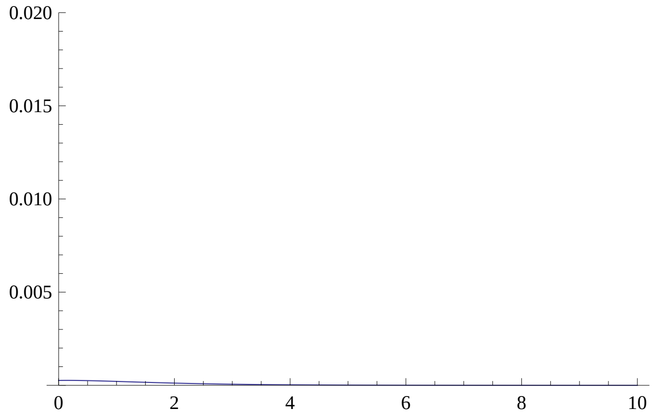


FIGURE 2.12. Error in Expected Value upon the Third Observation

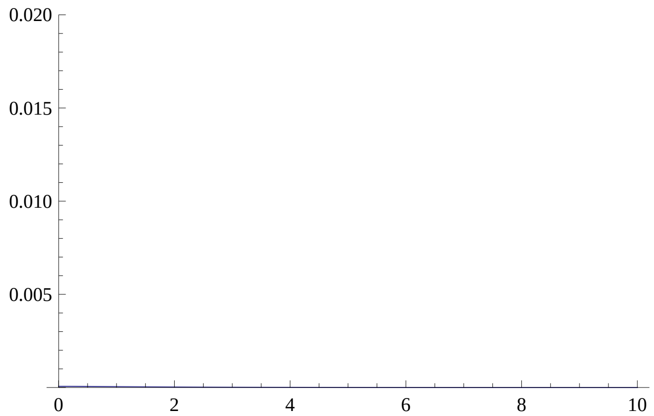


FIGURE 2.13. Error in Expected Value upon the Fourth Observation

2.5.2 Approximating the Expected Values of Evolutionary Systems

As eluded to in the previous section, we will now consider the same evolutionary system $T(s)$ with initial state given by $T(s)x$, where $x(r) = re^{-r}$. We assume that the probability density function $\pi(s) := 2(1-s)\chi_{[0,1]}(s)$ described the time upon which the n^{th} observation of the system is made. Therefore as we have previously shown the expected value (state) of the system upon the n^{th} observation is given by

$$E(n)x = \int_0^\infty T(s)x\,\pi_n(s)\,ds = \int_0^\infty T(s)x\,\pi^{\star n}(s)\,ds.$$

We now rely strictly upon the inversion methods previously described to illustrate the expected states of the system.

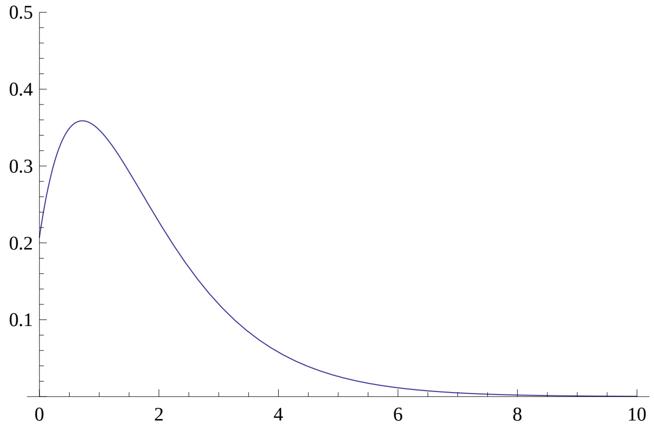


FIGURE 2.14. Expected Value upon First Observation

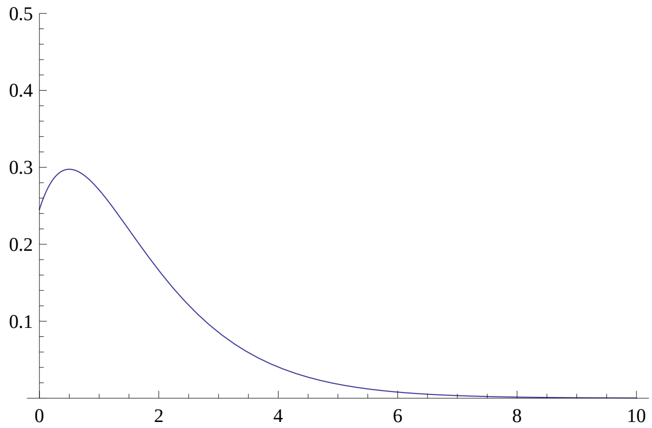


FIGURE 2.15. Expected Value upon Second Observation

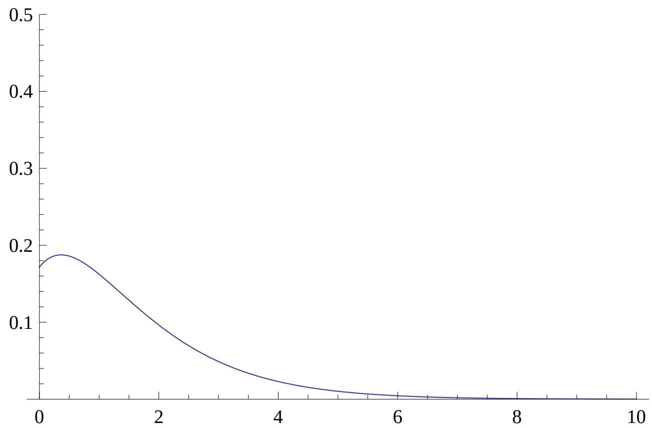


FIGURE 2.16. Expected Value upon the Third Observation

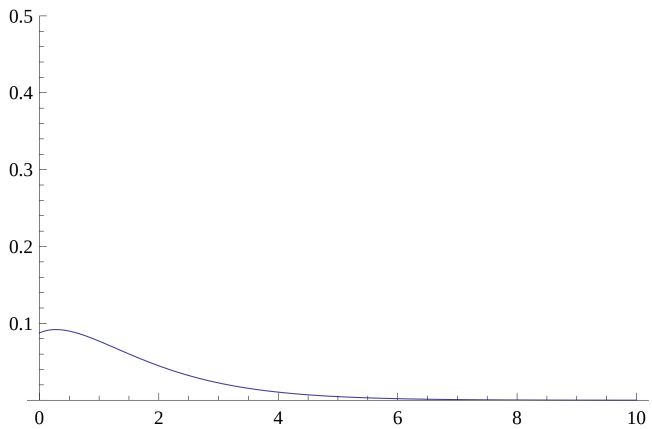


FIGURE 2.17. Expected Value upon the Fourth Observation

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Vita

Kevin W. Zito was born in 1979 in Baton Rouge, Louisiana. He finished his undergraduate studies at Louisiana State University in pure mathematics in May 2001. He earned a master of education degree in secondary mathematics education from Louisiana State University in August 2002. In August 2002 he began studies at the Deparment of Mathematics at Louisiana State University to persue his dreams of becoming a pure mathemtatician. He received a master of science degree in mathematics in January 2005. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2009.